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General Solutions of Relativistic Wave Equations

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General solutions of relativistic wave equations are studied in terms of functions on the Lorentz group. A close relationship between hyperspherical functions and matrix elements of irreducible representations of the Lorentz group is established. A generalization of the Gel'fand–Yaglom theory for higher-spin equations is given. A two-dimensional complex sphere is associated with each point of Minkowski spacetime. The separation of variables in a general relativistically invariant system is obtained via the hyperspherical functions defined on the surface of the two-dimensional complex sphere. In virtue of this the wave functions are represented in the form of series in hyperspherical functions. Such a description allows one to consider all the physical fields on an equal footing. General solutions of the Dirac and Weyl equations, and also the Maxwell equations in the Majorana–Oppenheimer form, are given in terms of functions on the Lorentz group.

KEY WORDS: relativistic wave equations; Lorentz group; Gel'fand–Yaglom chains; hyperspherical functions; two-dimensional complex sphere.

1. INTRODUCTION

Traditionally, equations of motion play a basic role in physics. Relativistic wave equations (RWE) play the same role in quantum field theory. The wave function (main object of quantum theory) is a solution of RWE. For that reason all textbooks on quantum field theory began with a brief introduction to RWE. As a rule, solutions of the Dirac equations are represented by plane-wave solutions (see for example (Schweber, 1961), and also many other textbooks). However, the plane-wave solutions are strongly degenerate. These solutions do not contain parameters of the Lorentz group and weakly reflect a relativistic nature of the wave function described by the Dirac equations.

In the present paper general solutions of RWE are given in the form of series of hyperspherical functions. In turn, matrix elements of irreducible representations of the Lorentz group are expressed via the hyperspherical functions

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(for more details see recent paper (Varlamov, 2002b)). By this reason the wave function hardly depends on the parameters of the Lorentz group. Moreover, it allows to consider solutions of RWE as the functions on the Lorentz group.

A starting point of the research is an isomorphism $SL(2,\mathbb{C}) \sim \mathsf{complex}(SU(2))$, that is, the group $SL(2,\mathbb{C})$ is a complexification of $SU(2)$. The well-known Van der Waerden representation for the Lorentz group (Van der Waerden, 1932) is a direct consequence of this isomorphism.

The following important point of the present research is a generalization of the Gel'fand–Yaglom formalism. In accordance with the fundamental isomorphism $SL(2,\mathbb{C}) \sim \mathsf{complex}(SU(2))$, a general relativistically invariant system is obtained as the result of complexification of three-dimensional Gel'fand–Yaglom equations. Further, a correspondence of the obtained invariant system with four-dimensional Gel'fand–Yaglom equations is established via the mapping of complexified equations into a six-dimensional bivector space $\mathbb{R}^6$ associated with the each point of the Minkowski spacetime $\mathbb{R}^{1,3}$.

The following step is a separation of variables in the relativistically invariant system. At this point, all the variables are parameters of the Lorentz group. The main tool for separation of variables is a two-dimensional complex sphere (this sphere was firstly considered by Smorodinsky and Huszar at the study of helicity states (Smorodinsky and Huszar, 1970)). Hyperspherical functions, defined on the surface of the two-dimensional complex sphere, allow to separate variables in the general relativistically invariant system.

Group theoretical description of RWE allows to present all the physical fields on an equal footing. Namely, all these fields are the functions on the Lorentz group. For example, in the sections 4–6 solutions of the Dirac, Weyl, and Maxwell equations, which considered as the particular cases of the general relativistically invariant system, are represented via the hyperspherical series in terms of the functions on the Lorentz group.

2. LORENTZ GROUP AND HYPERSPHERICAL FUNCTIONS

2.1. Van der Waerden Representation of the Lorentz Group

Let $\mathfrak{g} \rightarrow T_{\mathfrak{g}}$ be an arbitrary linear representation of the proper orthochronous Lorentz group $\mathfrak{G}_+$ and let $\mathsf{A}_i(t) = T_{a_i(t)}$ be an infinitesimal operator corresponded the rotation $a_i(t) \in \mathfrak{G}_+$. Analogously, we have $\mathsf{B}_i(t) = T_{bi}(t)$, where $b_i(t) \in \mathfrak{G}_+$ is a hyperbolic rotation. The operators A_i and B_i satisfy the following commutation relations[2]:

[2]Denoting $\mathsf{I}^{23} = \mathsf{A}_1$, $\mathsf{I}^{31} = \mathsf{A}_2$, $\mathsf{I}^{12} = \mathsf{A}_3$, and $\mathsf{I}^{01} = \mathsf{B}_1$, $\mathsf{I}^{02} = \mathsf{B}_2$, $\mathsf{I}^{03} = \mathsf{B}_3$ we acn write the relations (1) in a more compact from:

$$[\mathsf{I}^{\mu\nu}, \mathsf{I}^{\lambda\rho}] = \delta_{\mu\rho}\mathsf{I}^{\lambda\rho} + \delta_{\nu\lambda}\mathsf{I}^{\mu\rho} - \delta_{\nu\rho}\mathsf{I}^{\mu\lambda} - \delta_{\mu\lambda}\mathsf{I}^{\nu\rho}.$$

$$
\left.\begin{array}{lll}
[\mathsf{A}_1, \mathsf{A}_2] = \mathsf{A}_3, & [\mathsf{A}_2, \mathsf{A}_3] = \mathsf{A}_1, & [\mathsf{A}_3, \mathsf{A}_1] = \mathsf{A}_2, \\
[\mathsf{B}_1, \mathsf{B}_2] = -\mathsf{A}_3, & [\mathsf{B}_2, \mathsf{B}_3] = -\mathsf{A}_1, & [\mathsf{B}_3, \mathsf{B}_1] = -\mathsf{A}_2, \\
[\mathsf{A}_1, \mathsf{B}_1] = 0, & [\mathsf{A}_2, \mathsf{B}_2] = 0, & [\mathsf{A}_3, \mathsf{B}_3] = 0, \\
[\mathsf{A}_1, \mathsf{B}_2] = \mathsf{B}_3, & [\mathsf{A}_1, \mathsf{B}_3] = -\mathsf{B}_2, & \\
[\mathsf{A}_2, \mathsf{B}_3] = \mathsf{B}_1, & [\mathsf{A}_2, \mathsf{B}_1] = -\mathsf{B}_3, & \\
[\mathsf{A}_3, \mathsf{B}_1] = \mathsf{B}_2, & [\mathsf{A}_3, \mathsf{B}_2] = -\mathsf{B}_1. &
\end{array}\right\} \tag{1}
$$

Let us consider the operators

$$
\mathsf{X}_k = \frac{1}{2}(\mathsf{A}_k + i\mathsf{B}_k), \qquad \mathsf{Y}_k = \frac{1}{2}(\mathsf{A}_k - i\mathsf{B}_k), \qquad (k = 1, 2, 3). \tag{2}
$$

Using the relations (1) we find that

$$
\begin{array}{lll}
[\mathsf{X}_1, \mathsf{X}_2] = \mathsf{X}_3, & [\mathsf{X}_2, \mathsf{X}_3] = \mathsf{X}_1, & [\mathsf{X}_2, \mathsf{X}_1] = \mathsf{X}_2, \\
[\mathsf{Y}_1, \mathsf{Y}_2] = \mathsf{Y}_3, & [\mathsf{Y}_2, \mathsf{Y}_3] = \mathsf{Y}_1, & [\mathsf{Y}_3, \mathsf{Y}_1] = \mathsf{Y}_2, \\
[\mathsf{X}_k, \mathsf{Y}_l] = 0, \quad (k, l = 1, 2, 3). & &
\end{array} \tag{3}
$$

Further, taking

$$
\left.\begin{array}{ll}
\mathsf{X}_+ = \mathsf{X}_1 + i\mathsf{X}_2, & \mathsf{X}_- = \mathsf{X}_1 - i\mathsf{X}_2, \\
\mathsf{Y}_+ = \mathsf{Y}_1 + i\mathsf{Y}_2, & \mathsf{Y}_- = \mathsf{Y}_1 - i\mathsf{Y}_2
\end{array}\right\} \tag{4}
$$

we see that in virtue of commutativity of the relations (3) a space of an irreducible finite-dimensional representation of the group $\mathfrak{G}_+$ can be stretched on the totality of $(2l+1)(2\dot{l}+1)$ basis vectors $|l, m; \dot{l}, \dot{m}\rangle$, where l, m, $\dot{l}$, $\dot{m}$ are integer or half-integer numbers, $-l \le m \le l$, $-\dot{l} \le \dot{m} \le \dot{l}$. Therefore,

$$
\begin{aligned}
\mathsf{X}_-|l, m; \dot{l}, \dot{m}\rangle &= \sqrt{(\dot{l} + \dot{m})(\dot{l} - \dot{m} + 1)}|l, m; \dot{l}, \dot{m} - 1\rangle (\dot{m} > -\dot{l}), \\
\mathsf{X}_+|l, m; \dot{l}, \dot{m}\rangle &= \sqrt{(\dot{l} - \dot{m})(\dot{l} + \dot{m} + 1)}|l, m; \dot{l}, \dot{m} + 1\rangle (\dot{m} < \dot{l}), \\
\mathsf{X}_3|l, m; \dot{l}, \dot{m}\rangle &= \dot{m}|l, m; \dot{l}, \dot{m}\rangle, \\
\mathsf{Y}_-|l, m; \dot{l}, \dot{m}\rangle &= \sqrt{(l + m)(l - m + 1)}|l, m - 1; \dot{l}, \dot{m}\rangle (\dot{m} > -l), \\
\mathsf{Y}_+|l, m; \dot{l}, \dot{m}\rangle &= \sqrt{(l - m)(l + m + 1)}|l, m + 1; \dot{l}, \dot{m}\rangle (m < l), \\
\mathsf{Y}_3|l, m; \dot{l}, \dot{m}\rangle &= m|l, m; \dot{l}, \dot{m}\rangle.
\end{aligned} \tag{5}
$$

From the relations (3) it follows that each of the sets of infinitisimal operators X and Y generates the group $SU(2)$ and these two groups commute with each other. Thus, from the relations (3) and (5) it follows that the group $\mathfrak{G}_+$, in essence, is equivalent to the group $SU(2) \otimes SU(2)$. In contrast to the Gel'fand–Naimark representation for the Lorentz group (Gel'fand *et al.*, 1963; Naimark, 1964) which does not find a broad application in physics, a representation (5) is a most useful in theoretical physics (see, for example, Akhiezer and Berestetskii, 1965; Rumer and Fet, 1977;

Ryder, 1985; Schweber, 1961). This representation for the Lorentz group was firstly given by Van der Waerden in his brilliant book (Ven der Waerden, 1932).

As known, a double covering of the proper orthochronous Lorentz group $\mathfrak{G}_+$, the group $SL(2,\mathbb{C})$, is isomorphic to the Clifford–Lipschitz group $\mathbf{Spin}_+(1,3)$, which, in its turn, is fully defined within a biquaternion algebra $\mathbb{C}_2$, since

$$\mathbf{Spin}_+(1,3) \simeq \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{C} : \det = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 1 \right\} = SL(2,\mathbb{C}).$$

Thus, a fundamental representation of the group $\mathfrak{G}_+$ is realized in a spinspace $\mathbb{S}_2$. The spinspace $\mathbb{S}_2$ is a complexification of the minimal left ideal of the algebra $\mathbb{C}_2$: $\mathbb{S}_2 = \mathbb{C} \otimes I_{2,0} = \mathbb{C} \otimes C\ell_{2,0}e_{20}$ or $\mathbb{S}_2 = \mathbb{C} \otimes I_{1,1} = \mathbb{C} \otimes C\ell_{1,1}e_{11}$ ($\mathbb{C} \otimes I_{0,2} = \mathbb{C} \otimes C\ell_{0,2}e_{02}$), where $C\ell_{p,q}(p+q=2)$ is a real subalgebra of $\mathbb{C}_2$, $I_{p,q}$ is the minimal left ideal of the algebra $C\ell_{p,q}$, e_{pq} is a primitive idempotent.

Further, let $\overset{*}{\mathbb{C}}_2$ be the biquaternion algebra, in which all the coefficients are complex conjugate to the coefficients of the algebra $\mathbb{C}_2$. The algebra $\overset{*}{\mathbb{C}}_2$ is obtained from $\mathbb{C}_2$ under action of the automorphism $\mathcal{A} \to \mathcal{A}^\star$ (involution) or the antiautomorphism $\mathcal{A} \to \tilde{\mathcal{A}}$ (reversal), where $\mathcal{A} \in \mathbb{C}_2$ (see (Varlamov, 1999, 2001)). Let us compose a tensor product of k algebras $\mathbb{C}_2$ and r algebras $\overset{*}{\mathbb{C}}_2$:

$$\mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \cdots \otimes \mathbb{C}_2 \otimes \overset{*}{\mathbb{C}}_2 \otimes \overset{*}{\mathbb{C}}_2 \otimes \cdots \otimes \overset{*}{\mathbb{C}}_2 \simeq \mathbb{C}_{2k} \otimes \overset{*}{\mathbb{C}}_{2r}. \tag{6}$$

The tensor product (6) induces a spinspace

$$\mathbb{S}_2 \otimes \mathbb{S}_2 \otimes \cdots \otimes \mathbb{S}_2 \otimes \dot{\mathbb{S}}_2 \otimes \dot{\mathbb{S}}_2 \otimes \cdots \otimes \dot{\mathbb{S}}_2 = \mathbb{S}_{2^{k+r}} \tag{7}$$

with "vectors" (spintensors) of the form

$$\xi^{\alpha_1\alpha_2\cdots\alpha_k\dot{\alpha}_1\dot{\alpha}_2\cdots\dot{\alpha}_r} = \sum \xi^{\alpha_1} \otimes \xi^{\alpha_2} \otimes \cdots \otimes \xi^{\alpha_k} \otimes \xi^{\dot{\alpha}_1} \otimes \xi^{\dot{\alpha}_2} \otimes \cdots \otimes \xi^{\dot{\alpha}_r}. \tag{8}$$

The full representation space $\mathbb{S}_{2^{k+r}}$ contains both symmetric and antisymmetric spintensors (8). Usually, at the definition of irreducible finite-dimensional representations of the Lorentz group physicists confined to a subspace of symmetric spintensors $\mathrm{Sym}(k,r) \subset \mathbb{S}_2^{k+r}$. Dimension of $\mathrm{Sym}(k,r)$ is equal to $(k+1)(r+1)$ or $(2l+1)(2l'+1)$ at $l = \frac{k}{2}$, $l' = \frac{r}{2}$. It is easy to see that the space $\mathrm{Sym}(k,r)$ is a space of Van der Waerden representation (5). The space $\mathrm{Sym}(k,r)$ can be considered as a space of polynomials

$$p(z_0, z_1, \overline{z}_0, \overline{z}_1) = \sum_{\substack{(\alpha_1,\cdots,\alpha_k) \\ (\dot{\alpha}_1,\cdots,\dot{\alpha}_r)}} \frac{1}{k!\,r!} a^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_r} z_{\alpha_1}\cdots z_{\alpha_k}\overline{z}_{\dot{\alpha}_1}\cdots\overline{z}_{\dot{\alpha}_r} \quad (\alpha_i, \dot{\alpha}_i = 0, 1), \tag{9}$$

where the numbers $a^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_r}$ are unaffected at the permutations of indices.

The infinitesimal operators A_i, defined on the symmetric representation spaces, have the form

$$\mathsf{A}_1 = -\frac{i}{2}\alpha^l_m \xi_{m-1} - \frac{i}{2}\alpha^l_{m+1}\xi_{m+1},$$

$$\mathsf{A}_2 = \frac{1}{2}\alpha^l_m \xi_{m-1} - \frac{1}{2}\alpha^l_{m+1}\xi_{m+1},$$
$$\mathsf{A}_3 = -im\xi_m, \tag{10}$$

where

$$\alpha^l_m = \sqrt{(l+m)(l-m+1)},$$

and ξ_i are vectors of a canonical basis in the representation space. Or, in the matrix notation:

$$\mathsf{A}^j_1 = -\frac{i}{2}\begin{bmatrix} 0 & \alpha_{-l_j+1} & 0 & \cdots & 0 & 0 \\ \alpha_{-l_j+1} & 0 & \alpha_{-l_j+2} & \cdots & 0 & 0 \\ 0 & \alpha_{-l_j+2} & 0 & \cdots & 0 & 0 \\ \hdotsfor{6} \\ \hdotsfor{6} \\ 0 & 0 & 0 & \cdots & 0 & \alpha_{l_j} \\ 0 & 0 & 0 & \cdots & \alpha_{l_j} & 0 \end{bmatrix} \tag{11}$$

$$\mathsf{A}^j_2 = \frac{1}{2}\begin{bmatrix} 0 & \alpha_{-l_j+1} & 0 & \cdots & 0 & 0 \\ -\alpha_{-l_j+1} & 0 & \alpha_{-l_j+2} & \cdots & 0 & 0 \\ 0 & \alpha_{-l_j+2} & 0 & \cdots & 0 & 0 \\ \hdotsfor{6} \\ \hdotsfor{6} \\ 0 & 0 & 0 & \cdots & 0 & \alpha_{l_j} \\ 0 & 0 & 0 & \cdots & -\alpha_{l_j} & 0 \end{bmatrix} \tag{12}$$

$$\mathsf{A}^j_3 = \begin{bmatrix} il_j & 0 & 0 & \cdots & 0 & 0 \\ 0 & i(l_j-1) & 0 & \cdots & 0 & 0 \\ 0 & 0 & i(l_j-2) & \cdots & 0 & 0 \\ \hdotsfor{6} \\ \hdotsfor{6} \\ 0 & 0 & 0 & \cdots & -i(l_j-1) & 0 \\ 0 & 0 & 0 & \cdots & 0 & -jl_j \end{bmatrix} \tag{13}$$

Further, for the operators $\mathsf{B}_i = i\mathsf{A}_i$ we have

$$\mathsf{B}_1 = \frac{1}{2}\alpha^l_m \xi_{m-1} + \frac{1}{2}\alpha^l_{m+1}\xi_{m+1},$$
$$\mathsf{B}_2 = \frac{i}{2}\alpha^l_m \xi_{m-1} - \frac{i}{2}\alpha^l_{m+1}\xi_{m+1},$$
$$\mathsf{B}_3 = m\xi_m. \tag{14}$$

Analogously, infinitesimal operators for conjugate representations, in dual representation spaces, are defined by following formulae

$$\tilde{\mathsf{A}}_1 = \frac{i}{2}\alpha^l_m\xi_{m-1} + \frac{i}{2}\alpha^l_{m+1}\xi_{m+1},$$

$$\tilde{\mathsf{A}}_2 = -\frac{1}{2}\alpha^l_m\xi_{m-1} + \frac{1}{2}\alpha^l_{m+1}\xi_{m+1},$$

$$\tilde{\mathsf{A}}_3 = im\xi_m, \tag{15}$$

$$\tilde{\mathsf{B}}_1 = -\frac{1}{2}\alpha^l_m\xi_{m-1} - \frac{1}{2}\alpha^l_{m+1}\xi_{m+1},$$

$$\tilde{\mathsf{B}}_2 = -\frac{i}{2}\alpha^l_m\xi_{m-1} + \frac{i}{2}\alpha^l_{m+1}\xi_{m+1},$$

$$\tilde{\mathsf{B}}_3 = -m\xi_m. \tag{16}$$

2.2. Matrix Elements of $SU(2)$ and $SL(2,\mathbb{C})$

One-parameter subgroups of $SU(2)$ are defined by the matrices

$$a_1(t) = \begin{pmatrix} \cos\frac{t}{2} & i\sin\frac{t}{2} \\ i\sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix},$$

$$a_2(t) = \begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix}, \quad a_3(t) = \begin{pmatrix} e^{\frac{it}{2}} & 0 \\ 0 & e^{-\frac{it}{2}} \end{pmatrix}. \tag{17}$$

An arbitrary matrix $u \in SU(2)$ written via Euler angles has a form

$$u = \begin{pmatrix} \alpha & \beta \\ -\overset{*}{\beta} & \overset{*}{\alpha} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2}e^{\frac{i(\varphi+\psi)}{2}} & i\sin\frac{\theta}{2}e^{\frac{i(\varphi-\psi)}{2}} \\ i\sin\frac{\theta}{2}e^{\frac{i(\psi-\varphi)}{2}} & \cos\frac{\theta}{2}e^{-\frac{i(\varphi+\psi)}{2}} \end{pmatrix}, \tag{18}$$

where $0 \le \varphi < 2\pi$, $0 < \theta < \pi$, $-2\pi \le \psi < 2\pi$, $\det u = 1$. Hence it follows that $|\alpha| =$
$\cos\frac{\theta}{2}$, and $|\beta| = \sin\frac{\theta}{2}$ and

$$\cos\theta = 2|\alpha|^2 - 1, \tag{19}$$

$$e^{i\varphi} = -\frac{\alpha\beta_i}{|\alpha||\beta|}, \tag{20}$$

$$e^{\frac{i\varphi}{2}} = \frac{\alpha e^{-\frac{i\varphi}{2}}}{|\alpha|} \tag{21}$$

Diagonal matrices

$$\begin{pmatrix} e^{\frac{i\varphi}{2}} & 0 \\ 0 & e^{-\frac{i\varphi}{2}} \end{pmatrix}$$

form one-parameter subgroup in the group $SU(2)$. Therefore, each matrix $u \in SU(2)$ belongs to a bilateral adjacency class contained the matrix

$$\begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

The matrix element $t^l_{mn} = e^{-i(m\varphi+n\psi)}\langle T_l(\theta)\psi_n, \psi_m\rangle$ of the group $SU(2)$ in the polynomial basis

$$\psi_n(\zeta) = \frac{\zeta^{l-n}}{\sqrt{\Gamma(l-n+1)\Gamma(l+n+1)}} \qquad -l \le n \le l,$$

where

$$T_l(\theta)\psi(\zeta) = \left(i\sin\frac{\theta}{2}\zeta + \cos\frac{\theta}{2}\right)^{2l}\psi\left(\frac{\cos\frac{\theta}{2}\zeta + i\sin\frac{\theta}{2}}{i\sin\frac{\theta}{2}\zeta + \cos\frac{\theta}{2}}\right),$$

has a form

$$\begin{aligned} t^l_{mn}(g) &= e^{-i(m\varphi+n\psi)}\langle T_l(\theta)\psi_n, \psi_m\rangle \\ &= \frac{e^{-i(m\varphi+n\psi)}\langle T_l(\theta)\zeta^{l-n}\zeta^{l-m}\rangle}{\sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-n+1)\Gamma(l+n+1)}} \\ &= e^{-i(m\varphi+n\psi)} i^{m-n}\sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-n+1)\Gamma(l+n+1)} \\ &\times \cos^{2l}\frac{\theta}{2}\tan^{m-n}\frac{\theta}{2}\sum_{j=\max(0,n-m)}^{\min(l-n,l+n)} \\ &\times \frac{i^{2j}\tan^{2j}\frac{\theta}{2}}{\Gamma(j+1)\Gamma(l-m-j+1)\Gamma(l+n-j+1)\Gamma(m-n+j+1)}. \end{aligned} \tag{22}$$

Further, using the formula

$${}_2F_1\left(\begin{matrix}\alpha, \beta \\ \\ \gamma\end{matrix} \,\middle|\, z\right) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{k\ge 0}\frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)}\frac{z^k}{k!} \tag{23}$$

$$\begin{aligned} t^l_{mn}(g) &= \frac{i^{m-n}e^{-i(m\varphi+n\psi)}}{\Gamma(m-n+1)}\sqrt{\frac{\Gamma(l+m+1)\Gamma(l-n+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}} \\ &\times \cos^{2l}\frac{\theta}{2}\tan^{m-n}\frac{\theta}{2^2}F_1 \end{aligned}$$

$$\times \left(\begin{matrix} m-l+1, 1-l-n \\ m-n+1 \end{matrix} \middle| \; i^2 \tan^2 \frac{\theta}{2} \right), \tag{24}$$

where $m \geq n$. At $m < n$ in the right part of (24) it needs to replace m and n by $-m$ and $-n$, respectively. Since l, m, and n are finite numbers, then the hypergeometric series is interrupted.

Further, replacing in the one-parameter subgroups (17) the parameter t by $-it$ we obtain

$$b_1(t) = \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix}, \quad b_2(t) = \begin{pmatrix} \cosh \frac{t}{2} & i \sinh \frac{t}{2} \\ -i \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix},$$

$$b_3(t) = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}. \tag{25}$$

These subgroups correspond to hyperbolic rotations.

The group $SL(2, \mathbb{C})$ of all complex matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

of 2nd order with the determinant $\alpha\delta - \gamma\beta = 1$, is a *complexification* of the group $SU(2)$. The group $SU(2)$ is one of the real forms of $SL(2, \mathbb{C})$. The transition from $SU(2)$ to $SL(2, \mathbb{C})$ is realized via the complexification of three real parameters φ, θ, ψ (Euler angles). Let $\theta^c = \theta - i\tau$, $\varphi^c = \varphi - i\epsilon$; $\psi^c = \psi - i\varepsilon$ be complex Euler angles, where

$$0 \leq \mathrm{Re}\theta^c = \theta \leq \pi, \qquad -\infty < \mathrm{Im}\theta^c = \tau < +\infty,$$

$$0 \leq \mathrm{Re}\varphi^c = \varphi < 2\pi, \qquad -\infty < \mathrm{Im}\varphi^c = \epsilon < +\infty,$$

$$-2\pi \leq \mathrm{Re}\psi^c = \psi < 2\pi, \qquad -\infty < \mathrm{Im}\psi^c = \varepsilon < +\infty.$$

Replacing in (18) the angles φ, θ, ψ by the complex angles $\varphi^c, \theta^c, \psi^c$ we come to the following matrix

$$\mathfrak{g} = \begin{pmatrix} \cos \frac{\theta^c}{2} e^{\frac{i(\varphi^c+\psi^c)}{2}} & i \sin \frac{\theta^c}{2} e^{\frac{i(\varphi^c-\psi^c)}{2}} \\ i \sin \frac{\theta^c}{2} e^{\frac{i(\psi^c-\varphi^c)}{2}} & \cos \frac{\theta^c}{2} e^{-\frac{i(\varphi^c+\psi^c)}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \left[\cos \frac{\theta}{2} \cosh \frac{\tau}{2} + i \sin \frac{\theta}{2} \sinh \frac{\tau}{2}\right] e^{\frac{\epsilon+\varepsilon+i(\varphi+\psi)}{2}} & \left[\cos \frac{\theta}{2} \sinh \frac{\tau}{2} + i \sin \frac{\theta}{2} \cosh \frac{\tau}{2}\right] e^{\frac{\epsilon-\varepsilon+i(\varphi-\psi)}{2}} \\ \left[\cos \frac{\theta}{2} \sinh \frac{\tau}{2} + i \sin \frac{\theta}{2} \cosh \frac{\tau}{2}\right] e^{\frac{\epsilon-\varepsilon+i(\varphi+\psi)}{2}} & \left[\cos \frac{\theta}{2} \cosh \frac{\tau}{2} + i \sin \frac{\theta}{2} \sinh \frac{\tau}{2}\right] e^{\frac{-\epsilon-\varepsilon-i(\varphi+\psi)}{2}} \end{pmatrix}, \tag{26}$$

since

$$\cos\frac{1}{2}(\theta - i\tau) = \cos\frac{\theta}{2}\cosh\frac{\tau}{2} + i\sin\frac{\theta}{2}\sinh\frac{\tau}{2}, \quad \text{and} \quad \sinh\frac{1}{2}(\theta - i\tau) = \sin\frac{\theta}{2}\cosh\frac{\tau}{2} - i\cos\frac{\theta}{2}\sinh\frac{\tau}{2}.$$

It is easy to verify that the matrix (26) coincides with a matrix of the fundamental representation of the group $SL(2,\mathbb{C})$ (in Euler parametrization):

$$\mathfrak{g}(\varphi,\epsilon,\theta,\tau,\psi,\varepsilon) = \begin{pmatrix} e^{i\frac{\varphi}{2}} & 0 \\ 0 & e^{-i\frac{\varphi}{2}} \end{pmatrix} \begin{pmatrix} e^{\frac{\epsilon}{2}} & 0 \\ 0 & e^{-\frac{\epsilon}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \times \begin{pmatrix} \cosh\frac{\tau}{2} & \sinh\frac{\tau}{2} \\ \sinh\frac{\tau}{2} & \cosh\frac{\tau}{2} \end{pmatrix} \begin{pmatrix} e^{i\frac{\psi}{2}} & 0 \\ 0 & e^{-i\frac{\psi}{2}} \end{pmatrix} \begin{pmatrix} e^{\frac{\epsilon}{2}} & 0 \\ 0 & e^{-\frac{\epsilon}{2}} \end{pmatrix}. \quad (27)$$

The matrix element $t^l_{mn} = e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}\langle T_l(\theta,\tau)\psi_\lambda,\psi_\lambda\rangle$ of the finite-dimensional representation of $SL(2,\mathbb{C})$ at $l = \dot{l}$ in the polynomial basis

$$\psi_\lambda(\zeta,\overline{\zeta}) = \frac{\zeta^{l-n}\overline{\zeta}^{l-m}}{\sqrt{\Gamma(l-n+1)\Gamma(l+n+1)\Gamma(l-m+1)\Gamma(l+m+1)}}$$

has a form

$$\begin{aligned} t^l_{mn}(\mathfrak{g}) &= e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)} Z^l_{mn} = e^{-m(\epsilon+\varphi)-n(\varepsilon+i\psi)} \\ &\times \sum_{k=-l}^{l} i^{m-k}\sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-k+1)\Gamma(l+k+1)} \\ &\times \cos^{2l}\frac{\theta}{2}\tan^{m-k}\frac{\theta}{2} \\ &\times \sum_{j=\max(0,k-m)}^{\min(l-m,l+k)} \frac{i^{2j}\tan^{2j}\frac{\theta}{2}}{\Gamma(j+1)\Gamma(l-m-j+1)\Gamma(l+k-j+1)\Gamma(m-k+j+1)} \\ &\times \sqrt{\Gamma(l-n+1)\Gamma(l+n+1)\Gamma(l-k+1)\Gamma(l+k+1)}\cosh^{2l}\frac{\tau}{2}\tanh^{n-k}\frac{\tau}{2} \\ &\times \sum_{s=\max(0,k-n)}^{\min(l-n,l+k)} \frac{\tanh^{2s}\frac{\tau}{2}}{\Gamma(s+1)\Gamma(l-n-s+1)\Gamma(l+k-s+1)\Gamma(n-k+s+1)}. \end{aligned} \quad (28)$$

We will call the functions Z^l_{mn} in (28) as *hyperspherical functions*. Using (23) we can write the hyperspherical functions Z^l_{mn} via the hypergeometric series:

$$\mathsf{Z}^l_{mn} = \cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum_{k=-l}^{l} i^{m-k}\tan^{m-k}\frac{\theta}{2}\tanh^{n-k}\frac{\tau}{2}$$
$$\times {}_2F_1\left(\begin{matrix} m-l+1, 1-l-k \\ m-k+1 \end{matrix} \,\middle|\, i^2\tan^2\tfrac{\theta}{2}\right)$$
$$\times {}_2F_1\left(\begin{matrix} n-l+1, 1-l-k \\ n-k+1 \end{matrix} \,\middle|\, \tanh^2\tfrac{\tau}{2}\right). \qquad (29)$$

Therefore, matrix elements can be expressed by means of the function (*a generalized hyperspherical function*)

$$\mathfrak{M}^l_{mn}(\mathfrak{g}) = e^{-m(\epsilon+i\varphi)} Z^l_{mn} e^{-n(\epsilon+i\psi)}, \qquad (30)$$

where

$$\mathsf{Z}^l_{mn} = \sum_{k=-l}^{l} P^l_{mk}(\cos\theta)\mathfrak{P}^l_{kn}(\cosh\tau), \qquad (31)$$

here $P^l_{mn}(\cos\theta)$ is a generalized spherical function on the group $SU(2)$ (see Gel'fand *et al.*, 1963), and $\mathcal{B}^l_{mn}$ is an analog of the generalized spherical function for the group $QU(2)$ (so-called Jacobi function (Vilenkin, 1968)). $QU(2)$ is a group of quasiunitary unimodular matrices of second order. As well as the group $SU(2)$, the group $QU(2)$ is one of the real forms of $SL(2, \mathbb{C})$ ($QU(2)$ is noncompact).

Infinitesimal operators of the group $\mathfrak{G}_+$ can be expressed via the Euler angles as follows (for detailed deriving of these formulae see (Varlamov, 2002b))

$$\mathsf{A}_1 = \cos\psi^c\frac{\partial}{\partial\theta} + \frac{\sin\psi^c}{\sin\theta^c}\frac{\partial}{\partial\varphi} - \cot\theta^c\sin\psi^c\frac{\partial}{\partial\psi}, \qquad (32)$$

$$\mathsf{A}_2 = -\sin\psi^c\frac{\partial}{\partial\theta} + \frac{\cos\psi^c}{\sin\theta^c}\frac{\partial}{\partial\varphi} - \cot\theta^c\cos\psi^c\frac{\partial}{\partial\psi}, \qquad (33)$$

$$\mathsf{A}_3 = \frac{\partial}{\partial\psi}, \qquad (34)$$

$$\mathsf{B}_1 = \cos\psi^c\frac{\partial}{\partial\tau} + \frac{\sin\psi^c}{\sin\theta^c}\frac{\partial}{\partial\epsilon} - \cot\theta^c\sin\psi^c\frac{\partial}{\partial\varepsilon}, \qquad (35)$$

$$\mathsf{B}_2 = -\sin\psi^c\frac{\partial}{\partial\tau} + \frac{\cos\psi^c}{\sin\theta^c}\frac{\partial}{\partial\epsilon} - \cot\theta^c\cos\psi^c\frac{\partial}{\partial\varepsilon}, \qquad (36)$$

$$\mathsf{B}_3 = \frac{\partial}{\partial\varepsilon}. \qquad (37)$$

It is easy to verify that operators A_i, B_i, defined by the formulae (32)–(37), are satisfy the commutation relations (1).

2.3. Casimir Operators and Differential Equations for Hyperspherical Functions

Taking into account the expressions (32)–(37), we can write the operators (2) in the form

$$\mathsf{X}_1 = \cos\psi^c \frac{\partial}{\partial\theta^c} + \frac{\sin\psi^c}{\sin\theta^c}\frac{\partial}{\partial\varphi^c} - \cot\theta^c \sin\psi^c \frac{\partial}{\partial\psi^c}, \tag{38}$$

$$\mathsf{X}_2 = -\sin\psi^c \frac{\partial}{\partial\theta^c} + \frac{\cos\psi^c}{\sin\theta^c}\frac{\partial}{\partial\varpi^c} - \cot\theta^c \cos\psi^c \frac{\partial}{\partial\psi^c}, \tag{39}$$

$$\mathsf{X}_3 = \frac{\partial}{\partial\psi^c}, \tag{40}$$

$$\mathsf{Y}_1 = \cos\dot{\psi}^c \frac{\partial}{\partial\dot{\theta}^c} + \frac{\sin\dot{\psi}^c}{\sin\dot{\theta}^c}\frac{\partial}{\partial\dot{\varphi}^c} - \cot\dot{\theta}^c \sin\dot{\psi}^c \frac{\partial}{\partial\dot{\psi}^c}, \tag{41}$$

$$\mathsf{Y}_2 = -\sin\dot{\psi}^c \frac{\partial}{\partial\dot{\theta}^c} + \frac{\cos\dot{\psi}^c}{\sin\dot{\theta}^c}\frac{\partial}{\partial\dot{\varphi}^c} - \cot\dot{\theta}^c \cos\dot{\psi}^c \frac{\partial}{\partial\dot{\psi}^c}, \tag{42}$$

$$\mathsf{Y}_3 = \frac{\partial}{\partial\dot{\psi}^c}, \tag{43}$$

where

$$\frac{\partial}{\partial\theta^c} = \frac{1}{2}\left(\frac{\partial}{\partial\theta} + i\frac{\partial}{\partial\tau}\right), \quad \frac{\partial}{\partial\varphi^2} = \frac{1}{2}\left(\frac{\partial}{\partial\varphi} + i\frac{\partial}{\partial\epsilon}\right), \quad \frac{\partial}{\partial\psi^c} = \frac{1}{2}\left(\frac{\partial}{\partial\psi} + i\frac{\partial}{\partial\varepsilon}\right),$$
$$\frac{\partial}{\partial\dot{\theta}^c} = \frac{1}{2}\left(\frac{\partial}{\partial\theta} - i\frac{\partial}{\partial\tau}\right), \quad \frac{\partial}{\partial\dot{\varphi}^2} = \frac{1}{2}\left(\frac{\partial}{\partial\varphi} - i\frac{\partial}{\partial\epsilon}\right), \quad \frac{\partial}{\partial\dot{\psi}^c} = \frac{1}{2}\left(\frac{\partial}{\partial\psi} - i\frac{\partial}{\partial\varepsilon}\right),$$

As known, for the Lorentz group there are two independent Casimir operators

$$\mathsf{X}^2 = \mathsf{X}_1^2 + \mathsf{X}_2^2 + \mathsf{X}_3^2 = \frac{1}{4}(\mathsf{A}^2 - \mathsf{B}^2 + 2i\mathsf{AB}),$$
$$\mathsf{Y}^2 = \mathsf{Y}_1^2 + \mathsf{Y}_2^2 + \mathsf{Y}_3^2 = \frac{1}{4}(\mathsf{A}^2 - \mathsf{B}^2 + 2i\mathsf{AB}). \tag{44}$$

Substituting (38)–(43) into (44) we obtain in Euler parametrization for the Casimir operators the following expressions

$$\mathsf{X}^2 = \frac{\partial^2}{\partial\theta^{c2}} + \cot\theta^c\frac{\partial}{\partial\theta^c} + \frac{1}{\sin^2\theta^c}\left[\frac{\partial^2}{\partial\varphi^{c2}} - 2\cos\theta^c\frac{\partial}{\partial\varphi^c}\frac{\partial}{\partial\psi^c} + \frac{\partial^2}{\partial\psi^{c2}}\right],$$
$$\mathsf{Y}^2 = \frac{\partial^2}{\partial\dot{\theta}^{c2}} + \cot\dot{\theta}^c\frac{\partial}{\partial\dot{\theta}^c} + \frac{1}{\sin^2\dot{\theta}^c}\left[\frac{\partial^2}{\partial\dot{\varphi}^{c2}} - 2\cos\dot{\theta}^c\frac{\partial}{\partial\dot{\varphi}^c}\frac{\partial}{\partial\dot{\psi}^c} + \frac{\partial^2}{\partial\dot{\psi}^{c2}}\right]. \quad (45)$$

Matrix elements of unitary irreducible representations of the Lorentz group are eigenfunctions of the operators (45):

$$[\mathsf{X}^2 + l(l+1)]\mathfrak{M}^l_{mn}(\varphi^c, \theta^c, \psi^c) = 0,$$
$$[\mathsf{Y}^2 + \dot{l}(\dot{l}+1)]\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\varphi}^c, \dot{\theta}^c, \dot{\psi}^c) = 0, \quad (46)$$

where

$$\mathfrak{M}^l_{mn}(\varphi^c, \theta^c, \psi^c) = e^{-i(m\varphi^c + n\psi^c)}\mathsf{Z}^l_{mn}(\theta^c),$$
$$\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\varphi}^c, \dot{\theta}^c, \dot{\psi}^c) = e^{-i(\dot{m}\dot{\varphi}^c + \dot{n}\dot{\varphi}^c)}\mathsf{Z}^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\theta}^c). \quad (47)$$

Substituting the hyperspherical functions (47) into (46) and taking into account the operators (45) we obtain

$$\left[\frac{d^2}{d\theta^{c2}} + \cot\theta^c\frac{d}{d\theta^c} - \frac{m^2 + n^2 - 2mn\cos\theta^c}{\sin^2\theta^c} + l(l+1)\right]\mathsf{Z}^l_{mn}(\theta^c) = 0,$$
$$\left[\frac{d^2}{d\dot{\theta}^{c2}} + \cot\dot{\theta}^c\frac{d}{d\dot{\theta}^c} - \frac{\dot{m}^2 + \dot{n}^2 - 2\dot{m}\dot{n}\cos\dot{\theta}^c}{\sin^2\dot{\theta}^c} + \dot{l}(\dot{l}+1)\right]\mathsf{Z}^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\theta}^c) = 0.$$

Finally, after substitutions $z = \cos\theta^c$ and $\overset{*}{z} = \cos\dot{\theta}^c$ we come to the following differential equations

$$\left[(1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} - \frac{m^2 + n^2 - 3mnz}{1-z^2} + l(l+1)\right]\mathsf{Z}^l_{mn}(\arccos z) = 0,$$
$$\left[(1-\overset{*}{z}{}^2)\frac{d^2}{d\overset{*}{z}{}^2} - 2\overset{*}{z}\frac{d}{d\overset{*}{z}} - \frac{\dot{m}^2 + \dot{n}^2 - 3\dot{m}\dot{n}\overset{*}{z}}{1-\overset{*}{z}{}^2} + \dot{l}(\dot{l}+1)\right]\mathsf{Z}^{\dot{l}}_{\dot{m}\dot{n}}(\arccos\overset{*}{z}) = 0.$$

2.4. Recurrence Relations Between Hyperspherical Functions

Between generalized hyperspherical functions $\mathfrak{M}^l_{mn}$ (and also the hyperspherical functions Z^l_{mn}) there exists a wide variety of recurrence relations. Part of them relates the hyperspherical functions of one and the same order (with identical l), other part relates the functions of different orders (for more details see (Varlamov, 2002b)).

In virtue of the Van der Waerden representation (5) the recurrence formulae for the hyperspherical functions of one and the same order follow from the equalities

$$\mathsf{X}_{-}\dot{\mathfrak{M}}^{l}_{mn} = \dot{\alpha}_{n}\dot{\mathfrak{M}}^{l}_{m,n-1}, \qquad \mathsf{X}_{+}\dot{\mathfrak{M}}^{l}_{mn} = \dot{\alpha}_{n+1}\dot{\mathfrak{M}}^{l}_{m,n+1}, \tag{48}$$

$$\mathsf{Y}_{-}\mathfrak{M}^{l}_{mn} = \alpha_{n}\mathfrak{M}^{l}_{m,n-1}, \qquad \mathsf{Y}_{+}\mathfrak{M}^{l}_{mn} = \alpha_{n+1}\mathfrak{M}^{l}_{m,n+1}, \tag{49}$$

where

$$\dot{\alpha}_{n} = \sqrt{(\dot{l}+\dot{n})(\dot{l}-\dot{n}+1)}, \qquad \alpha_{n} = \sqrt{(l+n)(l-n+1)}$$

Using the formulae (32), (33), (35), and (36) we obtain

$$\mathsf{X}_{+} = \frac{e^{-i\psi^c}}{2}\left[\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta^c}\frac{\partial}{\partial\varphi} - i\cot\theta^c\frac{\partial}{\partial\psi} + i\frac{\partial}{\partial\tau} - \frac{1}{\sin\theta^c}\frac{\partial}{\partial\epsilon} + \cot\theta^c\frac{\partial}{\partial\varepsilon}\right], \tag{50}$$

$$\mathsf{X}_{-} = \frac{e^{-i\psi^c}}{2}\left[\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta^c}\frac{\partial}{\partial\varphi} + i\cot\theta^c\frac{\partial}{\partial\psi} + i\frac{\partial}{\partial\tau} + \frac{1}{\sin\theta^c}\frac{\partial}{\partial\epsilon} - \cot\theta^c\frac{\partial}{\partial\varepsilon}\right], \tag{51}$$

$$\mathsf{Y}_{+} = \frac{e^{-i\psi^c}}{2}\left[\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta^c}\frac{\partial}{\partial\varphi} - i\cot\theta^c\frac{\partial}{\partial\psi} - i\frac{\partial}{\partial\tau} + \frac{1}{\sin\theta^c}\frac{\partial}{\partial\epsilon} - \cot\theta^c\frac{\partial}{\partial\varepsilon}\right], \tag{52}$$

$$\mathsf{Y}_{-} = \frac{e^{-i\psi^c}}{2}\left[\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta^c}\frac{\partial}{\partial\varphi} + i\cot\theta^c\frac{\partial}{\partial\psi} - i\frac{\partial}{\partial\tau} - \frac{1}{\sin\theta^c}\frac{\partial}{\partial\epsilon} + \cot\theta^c\frac{\partial}{\partial\varepsilon}\right]. \tag{53}$$

Further, substituting the function $\mathfrak{M}^{l}_{mn} = e^{-m(\epsilon-\varphi)}Z^{l}_{mn}(\theta,\tau)e^{-n(\varepsilon-i\psi)}$ into the relations (48) and taking into account the operators (50) and (51) we find that

$$\frac{\partial\dot{Z}^{l}_{mn}}{\partial\theta} + i\frac{\partial\dot{Z}^{l}_{mn}}{\partial\tau} - \frac{2(m-n\cos\theta^c)}{\sin\theta^c}\dot{Z}^{l}_{mn} = 2\alpha'_{n}\dot{Z}^{l}_{m,n-1}, \tag{54}$$

$$\frac{\partial\dot{Z}^{l}_{mn}}{\partial\theta} + i\frac{\partial\dot{Z}^{l}_{mn}}{\partial\tau} + \frac{2(m-n\cos\theta^c)}{\sin\theta^c}\dot{Z}^{l}_{mn} = 2\alpha'_{n+1}\dot{Z}^{l}_{m,n+1}. \tag{55}$$

Since the functions $\dot{Z}^{l}_{mn}(\theta,\tau)$ are symmetric, that is $\dot{Z}^{l}_{mn}(\theta,\tau) = \dot{Z}^{l}_{nm}(\theta,\tau)$, then substituting $\dot{Z}^{l}_{nm}(\theta,\tau)$ in lieu of $\dot{Z}^{l}_{mn}$ into the formulae (54) and (55) and replacing

m by n, and n by m, we obtain

$$\frac{\partial Z^l_{mn}}{\partial\theta} + i\frac{\partial \mathsf{Z}^l_{mn}}{\partial\tau} - \frac{2(n - m\cos\theta^c)}{\sin\theta^c}\mathsf{Z}^l_{mn} = 2\alpha'_m Z^l_{m-1,n}, \tag{56}$$

$$\frac{\partial \mathsf{Z}^l_{mn}}{\partial\theta} + i\frac{\partial \mathsf{Z}^l_{mn}}{\partial\tau} + \frac{2(n - m\cos\theta^c)}{\sin\theta^c}\mathsf{Z}^l_{mn} = 2\alpha'_{m+1}\mathsf{Z}^l_{m+1,n}. \tag{57}$$

Analogously, for the relations (49) we have

$$\frac{\partial \dot{Z}^l_{mn}}{\partial\theta} - i\frac{\partial \dot{\mathsf{Z}}^l_{mn}}{\partial\tau} + \frac{2(n - m\cos\theta^c)}{\sin\theta^c}\dot{\mathsf{Z}}^l_{mn} = 2\alpha_m \dot{\mathsf{Z}}^l_{m-1,n}, \tag{58}$$

$$\frac{\partial \dot{\mathsf{Z}}^l_{mn}}{\partial\theta} - i\frac{\partial \dot{\mathsf{Z}}^l_{mn}}{\partial\tau} - \frac{2(n - m\cos\theta^c)}{\sin\theta^c}\dot{\mathsf{Z}}^l_{mn} = 2\alpha_{m+1}\dot{\mathsf{Z}}^l_{m+1,n}. \tag{59}$$

3. TWO-DIMENSIONAL COMPLEX SPHERE AND SEPARATION OF VARIABLES

3.1. Generalized Gel'fand–Yaglom Equations

In the three-dimensional Euclidear space $\mathbb{R}^3$ the functions $\psi_1(x_1, x_2, x_3)$, $\psi_2(x_1, x_2, x_3), \dots, \psi_N(x_1, x_2, x_3)$ satisfy the following system of invariant (under action of the group $SU(2)$) equations (Gel'fand *et al.*, 1963):

$$\sum_{i=1}^{3} L_i \frac{\partial\psi}{\partial x_i} + \kappa\psi = 0, \tag{60}$$

where L_i are n-dimensional matrices and κ is a number. Follows to the general group complexification $SL(2, \mathbb{C}) \sim \mathsf{complex}(SU(2))$ let us introduce a complex analog of the equations (60) in the three-dimensional complex space $\mathbb{C}^3$:

$$\sum_{i=1}^{3} \Lambda_i \frac{\partial\psi}{\partial x_i} - i\sum_{i=1}^{3} \Lambda_i \frac{\partial\psi}{\partial x_i^*} + \kappa^c\psi = 0,$$

$$\sum_{i=1}^{3} \Lambda_i^* \frac{\partial\dot{\psi}}{\partial\tilde{x}_i} + i\sum_{i=1}^{3} \Lambda_i^* \frac{\partial\dot{\psi}}{\partial\tilde{x}_i^*} + \dot{\kappa}^c\dot{\psi} = 0, \tag{61}$$

where Λ_i, Λ_i^* are n-dimensional matrices and κ^c is a complex number. We will call these equations as *generalized Gel'fand-Yaglom equations*.

The elements of the matrices Λ_i, Λ_i^* are

$$\Lambda_1 : \begin{cases} a^{\kappa'\kappa}_{l-1,l,m-1,m} &=& -\frac{c_{l-1,l}}{2}\sqrt{(l+m)(l+m-1)}, \\ a^{\kappa'\kappa}_{l,l,m-1,m} &=& \frac{c_{ll}}{2}\sqrt{(l+m)(l-m+1)}, \\ a^{\kappa'\kappa}_{l+1,l,m-1,m} &=& \frac{c_{l+1,l}}{2}\sqrt{(l-m+1)(l-m+2)}, \\ a^{\kappa'\kappa}_{l-1,l,m+1,m} &=& \frac{c_{l-1,l}}{2}\sqrt{(l-m)(l-m-1)}, \\ a^{\kappa'\kappa}_{l,l,m+1,m} &=& \frac{c_{ll}}{2}\sqrt{(l+m+1)(l-m)}, \\ a^{\kappa'\kappa}_{l+1,l,m+1,m} &=& -\frac{c_{l+1,l}}{2}\sqrt{(l+m+1)(l+m+2)}. \end{cases} \tag{62}$$

$$\Lambda_2 : \begin{cases} b^{\kappa'\kappa}_{l-1,l,m-1,m} &=& -\frac{ic_{l-1,l}}{2}\sqrt{(l+m)(l+m-1)}, \\ b^{\kappa'\kappa}_{l,l,m-1,m} &=& \frac{ic_{ll}}{2}\sqrt{(l+m)(l-m+1)}, \\ b^{\kappa'\kappa}_{l+1,l,m-1,m} &=& \frac{ic_{l+1,l}}{2}\sqrt{(l-m+1)(l-m+2)}, \\ b^{\kappa'\kappa}_{l-1,l,m+1,m} &=& -\frac{ic_{l-1,l}}{2}\sqrt{(l-m)(l-m-1)}, \\ b^{\kappa'\kappa}_{l,l,m+1,m} &=& -\frac{ic_{ll}}{2}\sqrt{(l+m+1)(l-m)}, \\ b^{\kappa'\kappa}_{l+1,l,m+1,m} &=& \frac{ic_{l+1,l}}{2}\sqrt{(l+m+1)(l+m+2)}. \end{cases} \tag{63}$$

$$\Lambda_3 : \begin{cases} c^{\kappa'\kappa}_{l-1,l,m} &=& c^{\kappa'\kappa}_{l-1,l}\sqrt{l^2-m^2}, \\ c^{\kappa'\kappa}_{l,l,m} &=& c^{\kappa'\kappa}_{ll}m, \\ c^{\kappa'\kappa}_{l+1,l,m} &=& c^{\kappa'\kappa}_{l+1,l}\sqrt{(l+1)^2-m^2}. \end{cases} \tag{64}$$

$$\Lambda_1^* : \begin{cases} d^{\dot{k}'\dot{k}}_{\dot{l}-1,\dot{l},\dot{m}-1,\dot{m}} &=& -\frac{c_{\dot{l}-1,\dot{l}}}{2}\sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m}-1)}, \\ d^{\dot{k}'\dot{k}}_{\dot{l},\dot{l},\dot{m}-1,\dot{m}} &=& \frac{c_{\dot{l}\dot{l}}}{2}\sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m}+1)}, \\ d^{\dot{k}'\dot{k}}_{\dot{l}-1,\dot{l},\dot{m}-1,\dot{m}} &=& \frac{c_{\dot{l}+1,\dot{l}}}{2}\sqrt{(\dot{l}-\dot{m}+1)(\dot{l}-\dot{m}+2)}, \\ d^{\dot{k}'\dot{k}}_{\dot{l}-1,\dot{l},\dot{m}+1,\dot{m}} &=& \frac{c_{\dot{l}-1,\dot{l}}}{2}\sqrt{(\dot{l}-\dot{m})(\dot{l}-\dot{m}-1)}, \\ d^{\dot{k}'\dot{k}}_{\dot{l},\dot{l},\dot{m}+1,\dot{m}} &=& \frac{c_{\dot{l}\dot{l}}}{2}\sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m})}, \\ d^{\dot{k}'\dot{k}}_{\dot{l}+1,\dot{l},\dot{m}+1,\dot{m}} &=& -\frac{c_{\dot{l}+1,\dot{l}}}{2}\sqrt{(\dot{l}+\dot{m}+1)(\dot{l}+\dot{m}+2)}. \end{cases} \tag{65}$$

$$\Lambda_2^*: \begin{cases} e^{\dot{k}'\dot{k}}_{\dot{l}-1,\dot{l},\dot{m}-1,\dot{m}} &= -\frac{ic_{\dot{l}-1,\dot{l}}}{2}\sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m}-1)}, \\ e^{\dot{k}'\dot{k}}_{\dot{l},\dot{l},\dot{m}-1,\dot{m}} &= \frac{ic_{\dot{l}\dot{l}}}{2}\sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m}+1)}, \\ e^{\dot{k}'\dot{k}}_{\dot{l}+1,\dot{l},\dot{m}-1,\dot{m}} &= \frac{ic_{\dot{l}+1,\dot{l}}}{2}\sqrt{(\dot{l}-\dot{m}+1)(\dot{l}-\dot{m}+2)}, \\ e^{\dot{k}'\dot{k}}_{\dot{l}-1,\dot{l},\dot{m}+1,\dot{m}} &= \frac{-ic_{\dot{l}-1,\dot{l}}}{2}\sqrt{(\dot{l}-\dot{m})(\dot{l}-\dot{m}-1)}, \\ e^{\dot{k}'\dot{k}}_{\dot{l},\dot{l},\dot{m}+1,\dot{m}} &= \frac{-ic_{\dot{l}\dot{l}}}{2}\sqrt{(\dot{l}+\dot{m}+1)(\dot{l}-\dot{m})}, \\ e^{\dot{k}'\dot{k}}_{\dot{l}+1,\dot{l},\dot{m}+1,\dot{m}} &= -\frac{ic_{\dot{l}+1,\dot{l}}}{2}\sqrt{(\dot{l}+\dot{m}+1)(\dot{l}+\dot{m}+2)}. \end{cases} \tag{66}$$

$$\Lambda_3^*: \begin{cases} f^{\dot{k}'\dot{k}}_{\dot{l}-1,\dot{l},\dot{m}} &= c^{\dot{k}'\dot{k}}_{\dot{l}-1,\dot{l}}\sqrt{\dot{l}^2-\dot{m}^2}, \\ f^{\dot{k}'\dot{k}}_{\dot{l}\dot{l},\dot{m}} &= c^{\dot{k}'\dot{k}}_{\dot{l}\dot{l}}\dot{m}, \\ f^{\dot{k}'\dot{k}}_{\dot{l}+1,\dot{l},\dot{m}} &= c^{\dot{k}'\dot{k}}_{\dot{l}+1,\dot{l}}\sqrt{(\dot{l}+1)^2-\dot{m}^2}. \end{cases} \tag{67}$$

The commutation relations between the matrices Λ_i, Λ_i^* and infinitesimal operators (10), (14)–(16) are

$$\begin{aligned} &[\mathsf{A}_1, \Lambda_1] = 0, \quad [\mathsf{A}_1, \Lambda_2] = \Lambda_3, \quad [\mathsf{A}_1, \Lambda_3], = -\Lambda_2, \\ &[\mathsf{A}_2, \Lambda_1] = -\Lambda_3, \quad [\mathsf{A}_2, \Lambda_2] = 0, \quad [\mathsf{A}_2, \Lambda_3] = \Lambda_1, \\ &[\mathsf{A}_3, \Lambda_1] = \Lambda_2, \quad [\mathsf{A}_3, \Lambda_2] = -\Lambda_1, \quad [\mathsf{A}_3, \Lambda_3] = 0. \end{aligned} \tag{68}$$

$$\begin{aligned} &[\mathsf{B}_1, \Lambda_1] = 0, \quad [\mathsf{B}_1, \Lambda_2] = i\Lambda_3, \quad [\mathsf{B}_1, \Lambda_3], = -i\Lambda_2, \\ &[\mathsf{B}_2, \Lambda_1] = -i\Lambda_3, \quad [\mathsf{B}_2, \Lambda_2] = 0, \quad [\mathsf{B}_2, \Lambda_3] = i\Lambda_1, \\ &[\mathsf{B}_3, \Lambda_1] = i\Lambda_2, \quad [\mathsf{B}_3, \Lambda_2] = -i\Lambda_1, \quad [\mathsf{B}_3, \Lambda_3] = 0. \end{aligned} \tag{69}$$

$$\begin{aligned} &[\tilde{\mathsf{A}}_1, \Lambda_1^*] = 0, \quad [\tilde{\mathsf{A}}_1, \Lambda_2^*] = -\Lambda_3^*, \quad [\tilde{\mathsf{A}}_1, \Lambda_3], = \Lambda_2^*, \\ &[\tilde{\mathsf{A}}_2, \Lambda_1^*] = \Lambda_3^*, \quad [\tilde{\mathsf{A}}_2, \Lambda_2^*] = 0, \quad [\tilde{\mathsf{A}}_2, \Lambda_3] = -\Lambda_1^*, \\ &[\tilde{\mathsf{A}}_3, \Lambda_1^*] = -\Lambda_2^*, \quad [\tilde{\mathsf{A}}_3, \Lambda_2^*] = \Lambda_1^*, \quad [\tilde{\mathsf{A}}_3, \Lambda_3^*] = 0. \end{aligned} \tag{70}$$

$$\begin{aligned} &[\tilde{\mathsf{B}}_1, \Lambda_1^*] = 0, \quad [\tilde{\mathsf{B}}_1^*, \Lambda_2^*] = -i\Lambda_3^*, \quad [\tilde{\mathsf{B}}_1, \Lambda_3^*], = i\Lambda_2^*, \\ &[\tilde{\mathsf{B}}_2, \Lambda_1^*] = i\Lambda_3^*, \quad [\tilde{\mathsf{B}}_2^*, \Lambda_2^*] = 0, \quad [\tilde{\mathsf{B}}_2, \Lambda_3^*] = -i\Lambda_1^*, \\ &[\tilde{\mathsf{B}}_3, \Lambda_1^*] = -i\Lambda_2^*, \quad [\tilde{\mathsf{B}}_3^*, \Lambda_2^*] = i\Lambda_1, \quad [\tilde{\mathsf{B}}_3^*, \Lambda_3^*] = 0. \end{aligned} \tag{71}$$

Let us establish now an important relationship between the complexification (61) and the four-dimensional Gel'fand–Yaglom formalism. As known, one of the most powerful higher spin formalisms is a Gel'fand–Yaglom approach (Gel'fand and Yaglom, 1948) based primarily on the representation theory of the Lorentz group. In contrast to the Bargmann–Wigner and Joos–Weinberg formalisms, the main advantage of the Gel'fand–Yaglom formalism lies in the fact that it admits naturally a Lagrangian formulation. Indeed, an initial point of this theory is the following lagrangian (Akhiezer and Berestetskii, 1965; Gel'fand *et al.*, 1963; Gel'fand and Yaglom, 1948):

$$\mathcal{L} = -\frac{1}{2}\left(\overline{\psi}\Gamma_\mu\frac{\partial\psi}{\partial x_\mu} - \frac{\partial\overline{\psi}}{\partial x_\mu}\Gamma_\mu\psi\right) - \kappa\overline{\psi}\psi, \tag{72}$$

where Γ_μ are n-dimensional matrices, n equals to the number of components of the wave function ψ. Varying independently the functions ψ and $\overline{\psi}$ one gets general Dirac-like (Gel'fand and Yaglom (1948)) equations

$$\begin{aligned}\Gamma_\mu\frac{\partial\psi}{\partial x_\mu} + \kappa\psi = 0,\\ \Gamma_\mu^T\frac{\partial\overline{\psi}}{\partial x_\mu} - \kappa\overline{\psi} = 0.\end{aligned} \tag{73}$$

As it shown in (Gel'fand *et al.*, 1963) (see also (Amar and Dozzio, 1972; Pletyukhov and Strazhev, 1983)) the matrix Γ_0 in 4D Gel'fand–Yaglom equations (73) can be written in the form

$$\Gamma_0 = \mathrm{diag}\left(C^0 \otimes I_1, C^1 \otimes I_3, \dots, C^s \otimes I_{2s+1}, \dots\right) \tag{74}$$

for integer spin and

$$\Gamma_0 = \mathrm{diag}\left(C^{\frac{1}{2}} \otimes I_2, C^{\frac{3}{2}} \otimes I_4, \dots, C^s \otimes I_{2s+1}, \dots\right) \tag{75}$$

for half-integer spin, where C^s is a spin block. If the spin block C^s has non-null roots, then the particle possesses the spin s. The spin block C^s in (74) and (75) consists of the elements $c^s_{\tau\tau'}$, where τ_{l_1,l_2} and $\tau_{l'_1,l'_2}$ are interlocking irreducible representations of the Lorentz group, that is, such representations, for which $l'_1 = l_1 \pm \frac{1}{2}, l'_2 = l_2 \pm \frac{1}{2}$. At this point the block C^s contains only the elements $c^s_{\tau\tau'}$, corresponding to such interlocking representations τ_{l_1,l_2}, $\tau_{l'_1,l'_2}$ which satisfy the conditions

$$|l_1 - l_2| \le s \le l_1 + L_2, \quad |l'_1 - l'_2| \le s \le l'_1 + l'_2.$$

The two most full schemes of the interlocking irreducible representations of the Lorentz group (Gel’fand–Yaglom chains) for integer (scheme (76)) and half-integer (scheme (77)) spins are

$$
\begin{array}{ccccccccccc}
 & & & & & & & & (s,0) & — & \cdots \\
 & & & & & & & & | & & \\
 & & & & & & & & \vdots & & \\
 & & & & & & & & | & & \\
 & & & & (2,0) & — & \cdots & — & \left(\frac{s+2}{2},\frac{s-2}{2}\right) & — & \cdots \\
 & & & & | & & & & | & & \\
 & & (1,0) & — & \left(\frac{3}{2},\frac{1}{2}\right) & — & \cdots & — & \left(\frac{s+1}{2},\frac{s-1}{2}\right) & — & \cdots \\
 & & | & & | & & & & | & & \\
(0,0) & — & \left(\frac{1}{2},\frac{1}{2}\right) & — & (1,1) & — & \cdots & — & \left(\frac{s}{2},\frac{s}{2}\right) & — & \cdots \\
 & & | & & & & & & & & \\
 & & (0,1) & — & \left(\frac{1}{2},\frac{3}{2}\right) & — & \cdots & — & \left(\frac{s-1}{2},\frac{s+1}{2}\right) & — & \cdots \\
 & & & & | & & & & | & & \\
 & & & & (0,2) & — & \cdots & — & \left(\frac{s-2}{2},\frac{s+2}{2}\right) & — & \cdots \\
 & & & & & & & & | & & \\
 & & & & & & & & \vdots & & \\
 & & & & & & & & | & & \\
 & & & & & & & & (0,s) & — & \cdots
\end{array}
\tag{76}
$$

$$
\begin{array}{ccccccccc}
 & & & & & & (s,0) & \text{———} & \cdots \\
 & & & & & & | & & \\
 & & & & & & \vdots & & \\
 & & & & & & | & & \\
 & & \left(\frac{3}{2},0\right) & \text{———} & \cdots & \text{—} & \left(\frac{2s+3}{4},\frac{2s-3}{4}\right) & \text{—} & \cdots \\
 & & | & & & & | & & \\
\left(\frac{1}{2},0\right) & \text{———} & \left(1,\frac{1}{2}\right) & \text{———} & \cdots & \text{—} & \left(\frac{2s+1}{4},\frac{2s-1}{4}\right) & \text{—} & \cdots \\
| & & | & & & & | & & \\
\left(0,\frac{1}{2}\right) & \text{———} & \left(\frac{1}{2},1\right) & \text{———} & \cdots & \text{—} & \left(\frac{2s-1}{4},\frac{2s+1}{4}\right) & \text{———} & \cdots \\
 & & | & & & & | & & \\
 & & \left(0,\frac{3}{2}\right) & \text{———} & \cdots & \text{—} & \left(\frac{2s-3}{4},\frac{2s+3}{4}\right) & \text{—} & \cdots \\
 & & & & & & | & & \\
 & & & & & & \vdots & & \\
 & & & & & & | & & \\
 & & & & & & (0,s) & \text{———} & \cdots
\end{array}
\tag{77}
$$

3.2. Bivector Space $\mathbb{R}^6$

We will establish a relationship between complex equations (61) and 4D Gel'fand–Yaglom formalism by means of the mapping of the equations (73) onto a bivector space $\mathbb{R}^6$.

Let $\mathbb{R}^{p,q}$ be the n-dimensional pseudo-Euclidean space, $p+q=n$. Let us evolve in $\mathbb{R}^{p,q}$ all the tensors satisfying the following two conditions: (1) a rank of the tensors is even; (2) covariant and contravariant indexes are divided into separate skewsymmetric pairs. Such tensors can be exemplified by bivectors (skewsymmetric tensors of the second rank). The set of all bivector tensor fields in $\mathbb{R}^{p,q}$ is called *a bivector set*, and its representation in a given point of $\mathbb{R}^{p,q}$ is called *a local bivector* set. In any tensor from the bivector set we take the each skewsymmetric pair $\alpha\beta$ as one collective index. At this point from the two possible pairs $\alpha\beta$ and $\beta\alpha$ we fix only one, for example, $\alpha\beta$. The number of all collective indexes is equal to

$N = \frac{n(n-1)}{2}$. In a given point, the bivector set of the space $\mathbb{R}^{p,q}$ with contravariant components defines in the collective indexes a vector set, the each vector of this set has N components. Identifying these vectors with the points of N-dimensional manifold, we see that it be an affine manifold E^N if and only if this manifold admits a Klein geometry with the group

$$\eta^{a'} = A^{a'}_{a}\eta^{a}, \qquad \eta^{a} = A^{a}_{a'}\eta^{a'},$$
$$\det A^{a'}_{a} \neq 0, \qquad A^{a}_{b'}A^{b'}_{c} = \delta^{a}_{c},$$

where

$$A^{a'}_{a} \longrightarrow A^{\alpha'}_{[\alpha}A^{\beta'}_{\beta]}.$$

Thus, any local bivector set of the space $\mathbb{R}^{p,q}(p+q=n)$ can be mapped onto the affine space E^N. Therefore, E^N related with the each point of the space $\mathbb{R}^{p,q}$. The space E^N is called *a bivector space*. It should be noted that the bivector space is a particular case of the most general mathematical construction called a Grassmannian manifold (manifold of m-dimensional planes of the affine space). In the case $m = 2$ the manifold of two-dimensional planes is isometric to the bivector space, and the Grassmann coordinates in this case are called Pluecker coordinates.

The metrization of the bivector space E^N is given by the formula (see (Petrov, 1969))

$$g_{ab} \to g_{\alpha\beta\gamma\delta} \equiv g_{\alpha\gamma}g_{\beta\gamma} - g_{\alpha\delta}g_{\beta\gamma}, \tag{78}$$

where $g_{\alpha\beta}$ is a metric tensor of the space $\mathbb{R}^{p,q}$, and the collective indexes are skewsymmetric pairs $\alpha\beta \to a$, $\gamma\delta \to b$. After introduction of g_{ab}, the bivector affine space E^N is transformed to a metric space $\mathbb{R}^N$.

In the case of Minkowski spacetime $\mathbb{R}^{1,3}$ with the metric tensor

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

in virtue of (78) we obtain for the bivector space $\mathbb{R}^6$:

$$g_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{79}$$

where the order of collective indexes in $\mathbb{R}^6$ is $23 \to 0$, $10 \to 1$, $20 \to 2$, $30 \to 3$, $31 \to 4$, $12 \to 5$. As it shown in (Kagan, 1926), the Lorentz transformations can be represented by linear transformations of the space $\mathbb{R}^6$. Let us write an invariance condition of the system (61). Let $\mathfrak{g} : x' = \mathfrak{g}^{-1}x$ be a transformation of the bivector space $\mathbb{R}^6$, that is $x' = \sum_{b=1}^{6} g_{ba}x_b$, where $x = (x_1, x_2, x_3, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ and g_{ba} is the metric tensor (79). We can write the tensor (79) in the form

$$g_{ab} = \begin{pmatrix} g_{ik}^{-} & \\ & g_{ik}^{-}, \end{pmatrix}$$

then $x' = \sum_{k=1}^{3} g_{ki}^{-} x_k$, $\tilde{x}' = \sum_{k=1}^{3} g_{ki}^{+} \tilde{x}_k$. Replacing ψ via $T_{\mathfrak{g}}^{-1}\psi'$ and differentiation on $x_k(\tilde{x}_k)$ by differentiation on $x_k'(\tilde{x}_k')$ via the formulae

$$\frac{\partial}{\partial x_k} = \sum g_{ik}^{-} \frac{\partial}{\partial x_i'}, \qquad \frac{\partial}{\partial \tilde{x}_k} = \sum g_{ik}^{+} \frac{\partial}{\partial \tilde{x}_i'}$$

we obtain

$$\sum_{i=1}^{3} \left[g_{i1}^{-} \Lambda_1 \frac{\partial \left(T_{\mathfrak{g}}^{-1}\psi'\right)}{\partial x_i'} + g_{i2}^{-} \Lambda_2 \frac{\partial \left(T_{\mathfrak{g}}^{-1}\psi'\right)}{\partial x_i'} + g_{i3}^{-} \Lambda_3 \frac{\partial \left(T_{\mathfrak{g}}^{-1}\psi'\right)}{\partial x_i'} \right.$$

$$\left. - i g_{i1}^{-} \Lambda_1 \frac{\partial \left(T_{\mathfrak{g}}^{-1}\psi'\right)}{\partial^* x_i'} - i g_{i2}^{+} \Lambda_2 \frac{\partial \left(T_{\mathfrak{g}}^{-1}\psi'\right)}{\partial^* x_i'} - i g_{i3}^{-} \Lambda_3 \frac{\partial \left(T_{\mathfrak{g}}^{-1}\psi'\right)}{\partial^* x_i'} \right] + \kappa^c T_{\mathfrak{g}}^{-1}\psi' = 0,$$

$$\sum_{i=1}^{3} \left[g_{i1}^{-} \Lambda_1^* \frac{\partial \left(\overset{*}{T}{}_{\mathfrak{g}}^{-1} \dot{\psi}' \right)}{\partial \tilde{x}_i'} + g_{i2}^{-} \Lambda_2^* \frac{\partial \left(\overset{*}{T}{}_{\mathfrak{g}}^{-1} \dot{\psi}' \right)}{\partial \tilde{x}_i'} + g_{i3}^{+} \Lambda_3^* \frac{\partial \left(\overset{*}{T}{}_{\mathfrak{g}}^{-1} \dot{\psi}' \right)}{\partial \tilde{x}_i'} \right.$$

$$\left. + i g_{i1}^{-} \Lambda_1^* \frac{\partial \left(\overset{*}{T}{}_{\mathfrak{g}}^{-1} \dot{\psi}' \right)}{\partial^* \tilde{x}_i'} + i g_{i2}^{-} \Lambda_2^* \frac{\partial \left(\overset{*}{T}{}_{\mathfrak{g}}^{-1} \dot{\psi}' \right)}{\partial^* \tilde{x}_i'} + i g_{i3}^{-} \Lambda_3^* \frac{\partial \left(\overset{*}{T}{}_{\mathfrak{g}}^{-1} \dot{\psi}' \right)}{\partial^* \tilde{x}_i'} \right]$$

$$+ \dot{\kappa}^c \overset{*}{T}{}_{\mathfrak{g}}^{-1} \dot{\psi}' = 0,$$

Or, since $T_{\mathfrak{g}}$ is a constant matrix, we have

$$\sum_i \left[g_{i1}^{-} \Lambda_1 T_{\mathfrak{g}}^{-1} \frac{\partial \psi'}{\partial x_i'} + g_{i2}^{-} \Lambda_2 T_{\mathfrak{g}}^{-1} \frac{\partial \psi'}{\partial x_i'} + g_{i3}^{-} \Lambda_3 T_{\mathfrak{g}}^{-1} \frac{\partial \psi'}{\partial x_i'} \right.$$

$$\left. - \; i g_{i1}^{-} \Lambda_1 T_{\mathfrak{g}}^{-1} \frac{\partial \psi'}{\partial x_i^{*\prime}} - i g_{i2}^{-} \Lambda_2 T_{\mathfrak{g}}^{-1} \frac{\partial \psi'}{\partial x_i^{*\prime}} - i g_{i3}^{-} \Lambda_3 T_{\mathfrak{g}}^{-1} \frac{\partial \psi'}{\partial x_i^{*\prime}} \right] + \kappa^c T_{\mathfrak{g}}^{-1}\psi' = 0,$$

$$\sum_i \left[g_{i1}^{+} \Lambda_1^* \overset{*}{T}{}_{\mathfrak{g}}^{-1} \frac{\partial \dot{\psi}'}{\partial \tilde{x}_i'} + g_{i2}^{+} \Lambda_2^* \overset{*}{T}{}_{\mathfrak{g}}^{-1} \frac{\partial \dot{\psi}'}{\partial \tilde{x}_i'} + g_{i3}^{+} \Lambda_3 \overset{*}{T}{}_{\mathfrak{g}}^{-1} \frac{\partial \dot{\psi}'}{\partial \tilde{x}_i'} \right.$$

$$+\ ig^+_{i1}\Lambda^*_1 \overset{*}{T}{}^{-1}_{\mathfrak{g}}\frac{\partial\dot{\psi}'}{\partial\tilde{x}^{*\prime}_i} + ig^+_{i2}\Lambda^*_2 \overset{*}{T}{}^{-1}_{\mathfrak{g}}\frac{\partial\dot{\psi}'}{\partial\tilde{x}^{*\prime}_i} + ig^+_{i3}\Lambda^*_3 \overset{*}{T}{}^{-1}_{\mathfrak{g}}\frac{\partial\dot{\psi}'}{\partial\tilde{x}^{*\prime}_i}\Bigg]$$

$$+\dot{\kappa}^c \overset{*}{T}{}^{-1}_{\mathfrak{g}}\dot{\psi}' = 0,$$

For coincidence of the latter system with (61) we must multiply this system by $T_{\mathfrak{g}}(\overset{*}{T}_{\mathfrak{g}})$ from the left:

$$\sum_i\sum_k g^-_{ik}T_{\mathfrak{g}}\Lambda_k T^{-1}_{\mathfrak{g}}\frac{\partial\psi'}{\partial x'_i} - i\sum_i\sum_k g^-_{ik}T_{\mathfrak{g}}\Lambda_k T^{-1}_{\mathfrak{g}}\frac{\partial\psi'}{\partial x^{*\prime}_i} + \kappa^c\psi' = 0,$$

$$\sum_i\sum_k g^+_{ik}\overset{*}{T}_{\mathfrak{g}}\Lambda^*_k \overset{*}{T}{}^{-1}_{\mathfrak{g}}\frac{\partial\dot{\psi}'}{\partial\tilde{x}'_i} + i\sum_i\sum_k g^+_{ik}\overset{*}{T}_{\mathfrak{g}}\Lambda^*_k \overset{*}{T}{}^{-1}_{\mathfrak{g}}\frac{\partial\dot{\psi}'}{\partial\tilde{x}^{*\prime}_i} + \dot{\kappa}^c\dot{\psi}' = 0,$$

The requirement of invariance means that for any transformation $\mathfrak{g}$ between the matrices $\Lambda_k(\Lambda^*_k)$ we must have the relations

$$\sum_k g^-_{ik}T_{\mathfrak{g}}\Lambda_k T^{-1}_{\mathfrak{g}} = \Lambda_i,$$

$$\sum_k g^+_{ik}\overset{*}{T}_{\mathfrak{g}}\Lambda^*_k \overset{*}{T}{}^{-1}_{\mathfrak{g}} = \Lambda^*_i. \tag{80}$$

In such a way, we see that the six-dimensional bivector space $\mathbb{R}^6$ is associated with the each point of the Minkowski spacetime $\mathbb{R}^{1,3}$. Using this fact we can establish now a relationship between the Λ-matrices of the equations (61) and Γ-matrices of the Gel'fand–Yaglom equations (73). We have here two essentially different cases:

1. The dimension of the Γ-matrices is even, $n \equiv 0 \pmod 2$. In this case the Γ-matrices have the form

$$\overset{i}{\Gamma}_{2^m} = \begin{pmatrix} \underset{i}{\Delta}_{2^{m-1}} & 0 \\ 0 & \underset{i}{\overset{*}{\Delta}}_{2^{m-1}} \end{pmatrix}, \quad \text{or} \quad \overset{i}{\Gamma}_{2^m} = \begin{pmatrix} 0 & \underset{i}{\Delta}_{2^{m-1}} \\ \underset{i}{\overset{*}{\Delta}}_{2^{m-1}} & 0 \end{pmatrix} \tag{81}$$

The Λ-matrices are expressed via (81) as follows

$$\underset{1}{\Lambda}_{2^{m-1}} = \underset{1}{\Delta}_{2^{m-1}}\underset{0}{\Delta}_{2^{m-1}}, \quad \underset{2}{\Lambda}_{2^{m-1}} = \underset{2}{\Delta}_{2^{m-1}}\underset{0}{\Delta}_{2^{m-1}}, \quad \underset{3}{\Lambda}_{2^{m-1}} = \underset{3}{\Delta}_{2^{m-1}}\underset{0}{\Delta}_{2^{m-1}},$$

$$\underset{4}{\Lambda}_{2^{m-1}} = \underset{2}{\overset{*}{\Delta}}_{2^{m-1}}\underset{3}{\overset{*}{\Delta}}_{2^{m-1}}, \quad \underset{5}{\Lambda}_{2^{m-1}} = \underset{3}{\overset{*}{\Delta}}_{2^{m-1}}\underset{1}{\overset{*}{\Delta}}_{2^{m-1}}, \quad \underset{6}{\Lambda}_{2^{m-1}} = \underset{1}{\overset{*}{\Delta}}_{2^{m-1}}\underset{2}{\overset{*}{\Delta}}_{2^{m-1}}, \tag{82}$$

2. $n \equiv 1 \pmod 2$. In this case a relationship between the Λ- and Λ-matrices is

$$\underset{1}{\Lambda}_n = \overset{1}{\Gamma}_n \overset{0}{\Gamma}_n, \quad \underset{2}{\Lambda}_n = \overset{2}{\Gamma}_n \overset{0}{\Gamma}_n, \quad \underset{3}{\Lambda}_n = \overset{3}{\Gamma}_n \overset{0}{\Gamma}_n,$$

$$\underset{4}{\Lambda}_n = \overset{2}{\Gamma}_n \overset{3}{\Gamma}_n, \quad \underset{5}{\Lambda}_n = \overset{3}{\Gamma}_n \overset{1}{\Gamma}_n, \quad \underset{6}{\Lambda}_n = \overset{1}{\Gamma}_n \overset{2}{\Gamma}_n. \tag{83}$$

3.3. Separation of Variables in Generalized Gel'fand–Yaglom Equations

Let us construct in $\mathbb{C}^3$ a two-dimensional complex sphere from the quantities $z_k = x_k + iyk$, $\overset{*}{z}_k = x_k - iyk$ as follows (see Fig. 1)

$$\mathrm{z}^2 = z_1^2 + z_2^2 + z_3^2 = \mathrm{x}^2 - \mathrm{y} + 2i\mathrm{xy} = r^2 \tag{84}$$

and its complex conjugate (dual) sphere

$$\overset{*}{\mathrm{z}}{}^2 = \overset{*}{z}{}_1^2 + \overset{*}{z}{}_2^2 + \overset{*}{z}{}_3^2 = \mathrm{x}^2 - \mathrm{y}^2 - 2i\mathrm{xy} = \overset{*}{r}{}^2. \tag{85}$$

It is well-known that both quantities $\mathrm{x}^2 - \mathrm{y}^2$, xy are invariant with respect to the Lorentz transformations, since a surface of the complex sphere is invariant

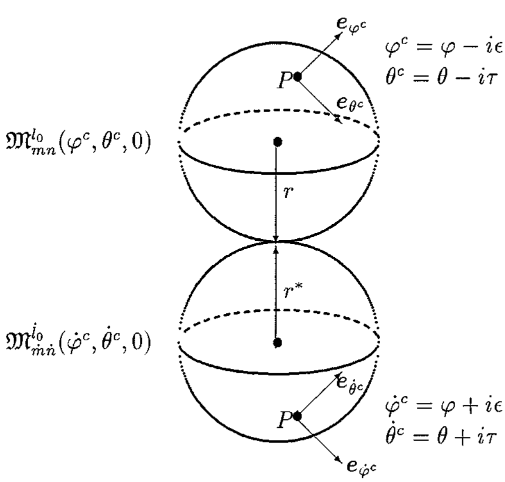

Fig. 1. Two-dimensional complex sphere $z_1^2 + z_2^2 + z_3^2 = r^2$ in three-dimensional complex space $\mathbb{C}^3$. The space $\mathbb{C}^3$ is isometric to the bivector space $\mathbb{R}^6$. The dual (complex conjugate) sphere $\overset{*}{z}{}_1^2 + \overset{*}{z}{}_2^2 + \overset{*}{z}{}_3^2 = \overset{*}{r}{}^2$ is a mirror image of the complex sphere with respect to the hyperplane. The hyperspherical functions $\mathfrak{M}^l_{mn}(\varphi^c, \theta^c, 0)$($\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\varphi}^c, \dot{\theta}^c, 0)$) are defined on the surface of the complex (dual) sphere.

(Casimir operators of the Lorentz group are constructed from such quantities, see also (44)). It is easy to see that three-dimensional complex space $\mathbb{C}^3$ is isometric to a real space $\mathbb{R}^{3,3}$ with a basis $\{i\mathbf{e}_1, i\mathbf{e}_2, i\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6$. At this point a metric tensor of $\mathbb{R}^{3,3}$ has the form (79). Hence it immediately follows that $\mathbb{C}^3$ is isometric to the bivector space $\mathbb{R}^6$.

Therefore, with the each point of the Minkowski spacetime $\mathbb{R}^{1,3}$ we can associate the two-dimensional complex sphere and its conjugate.

Let us introduce now hyperspherical coordinates on the surfaces of the complex and dual spheres

$$\begin{aligned} z_1 &= r\ \sin\theta^c\ \cos\varphi^c, \qquad z_1^* = r^*\ \sin\dot{\theta}^c\ \cos\dot{\varphi}^c, \\ z_2 &= r\ \sin\theta^c\ \sin\varphi^c, \qquad z_2^* = r^*\ \sin\dot{\theta}^c\ \sin\dot{\varphi}^c, \\ z_3 &= r\ \cos\theta^c, \qquad z_3^* = r^*\ \cos\dot{\theta}^c, \end{aligned} \tag{86}$$

where θ^c, φ^c are the complex Euler angles. Casimir operators on the two-dimensional complex sphere (correspondingly, on the dual sphere) have the form

$$\begin{aligned} \mathsf{X}^2 &= \frac{\partial^2}{\partial\theta^{c2}} + \cot\theta^c\frac{\partial}{\partial\theta^c} + \frac{1}{\sin^2\theta^c}\frac{\partial^2}{\partial\theta^{c2}} \\ \mathsf{Y}^2 &= \frac{\partial^2}{\partial\dot{\theta}^{c2}} + \cot\dot{\theta}^c\frac{\partial}{\partial\dot{\theta}^c} + \frac{1}{\sin^2\dot{\theta}^c}\frac{\partial^2}{\partial\dot{\theta}^{c2}}. \end{aligned} \tag{87}$$

Hyperspherical functions $\mathfrak{M}^l_{mn}(\varphi^c, \theta^c, 0)$ and $\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\varphi}^c, \dot{\theta}^c, 0)$, defined on the surface of the two-dimensional complex sphere, are eigenfunctions of the operators X^2 and Y^2:

$$\begin{aligned} &[\mathsf{X}^2 + l(l+1)]\mathfrak{M}^l_{mn}(\varphi^c, \theta^c, 0) = 0, \\ &[\mathsf{Y}^2 + \dot{l}(\dot{l}+1)]\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\varphi}^c, \dot{\theta}^c, 0) = 0 \end{aligned} \tag{88}$$

Substituting the functions $\mathfrak{M}^l_{mn}(\varphi^c, \theta^c, 0) = e^{-imn\varphi^c}Z^l_{mn}(\theta^c)$, $\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\varphi}^c, \dot{\theta}^c, 0) = e^{-i\dot{m}\dot{\varphi}^c}Z^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\theta}^c)$ into (88) and taking into account the operators (87) we obtain the following equations

$$\begin{aligned} &\left[\frac{d^2}{d\theta^{c2}} + \cot\theta^c\frac{d}{d\theta^c} - \frac{m^2}{\sin^2\theta^c} + l(l+1)\right]\mathsf{Z}^l_{mn}(\theta^c) = 0, \\ &\left[\frac{d^2}{d\dot{\theta}^{c2}} + \cot\dot{\theta}^c\frac{d}{d\dot{\theta}^c} - \frac{m^2}{\sin^2\dot{\theta}^c} + \dot{l}(\dot{l}+1)\right]\mathsf{Z}^{\dot{l}}_{mn}(\dot{\theta}^c) = 0, \end{aligned}$$

Or, introducing the substitutions $z = \cos\theta^c$, $\overset{*}{z} = \cos\dot{\theta}^c$ we find that

$$\left[(1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} - \frac{m^2}{1-z^2} + l(l+1)\right] Z^l_{mn}(\text{arc cos}\, z) = 0,$$

$$\left[(1-\overset{*}{z}{}^2)\frac{d^2}{d\overset{*}{z}{}^2}-2\overset{*}{z}\frac{d}{d\overset{*}{z}}-\frac{\dot{m}^2}{1-\overset{*}{z}{}^2}+\dot{l}(\dot{l}+1)\right]Z^{\dot{l}}_{\dot{m}\dot{n}}(\text{arc }\cos\overset{*}{z})=0,$$

The scalar product of the functions $\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\varphi}^c,\dot{\theta}^c,0)$ and $\mathfrak{M}^{l}_{mn}(\varphi^c,\theta^c,0)$, defined on the complex shere, is given by an expression

$$\int d\Omega\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}\mathfrak{M}^{l}_{mn}=\int d\varphi\, d\epsilon\, d\theta\, d\tau\ \sin\theta^c\ \sin\dot{\theta}^c\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\varphi}^c,\dot{\theta}^c,0)\mathfrak{M}^{l}_{mn}(\varphi^c,\theta^c,0)$$

with the integration limits

$$0\le\varphi<2\pi,\qquad -\infty<\epsilon<+\infty,$$
$$0\le\theta<\pi,\qquad -\infty<\tau<+\infty.$$

Let us show that a solution of the generalized Gel'fand–Yaglom equations (61), or the equations (73), can be found in the form of the series on generalized hyperspherical functions considered in the previous section.

With this end in view let us transform the system (61) as follows. First of all, let us define the derivatives $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial x_i^*}$ on the surface of the complex sphere (84) and write them in the hyperspherical coordinates (86) as

$$\frac{\partial}{\partial x_1}=-\frac{\sin\varphi^c}{r\ \sin\theta^c}\frac{\partial}{\partial\varphi}+\frac{\cos\varphi^c\cos\theta^c}{r}\frac{\partial}{\partial\theta}+\cos\varphi^c\ \sin\theta^c\frac{\partial}{\partial r},\tag{89}$$

$$\frac{\partial}{\partial x_2}=\frac{\cos\varphi^c}{r\ \sin\theta^c}\frac{\partial}{\partial\varphi}+\frac{\sin\varphi^c\cos\theta^c}{r}\frac{\partial}{\partial\theta}+\sin\varphi^c\ \sin\theta^c\frac{\partial}{\partial r},\tag{90}$$

$$\frac{\partial}{\partial x_3}=-\frac{\sin\theta^c}{r}\frac{\partial}{\partial\theta}+\cos\theta^c\frac{\partial}{\partial r}.\tag{91}$$

$$\frac{\partial}{\partial x_1^*}=i\frac{\partial}{\partial x_1}=-\frac{\sin\varphi^c}{r\ \sin\theta^c}\frac{\partial}{\partial\epsilon}+\frac{\cos\varphi^c\ \sin\theta^c}{r}\frac{\partial}{\partial\tau}+i\ \cos\varphi^c\ \sin\theta^c\frac{\partial}{\partial r},\tag{92}$$

$$\frac{\partial}{\partial x_2^*}=i\frac{\partial}{\partial x_2}=-\frac{\cos\varphi^c}{r\ \sin\theta^c}\frac{\partial}{\partial\epsilon}+\frac{\sin\varphi^c\ \cos\theta^c}{r}\frac{\partial}{\partial\tau}+i\ \sin\varphi^c\ \sin\theta^c\frac{\partial}{\partial r},\tag{93}$$

$$\frac{\partial}{\partial x_3^*}=i\frac{\partial}{\partial x_3}=-\frac{\sin\theta^c}{r}\frac{\partial}{\partial\tau}+i\ \cos\theta^c\frac{\partial}{\partial r}.\tag{94}$$

Analogously, on the surface of the dual sphere we have

$$\frac{\partial}{\partial\tilde{x}_1}=-\frac{\sin\dot{\varphi}^c}{r^*\ \sin\dot{\theta}^c}\frac{\partial}{\partial\varphi}+\frac{\cos\dot{\varphi}^c\ \cos\dot{\theta}^c}{r^*}\frac{\partial}{\partial\theta}+\cos\dot{\varphi}^c\ \sin\dot{\theta}^c\frac{\partial}{\partial r^*},\tag{95}$$

$$\frac{\partial}{\partial\tilde{x}_2}=-\frac{\cos\dot{\varphi}^c}{r^*\ \sin\dot{\theta}^c}\frac{\partial}{\partial\varphi}+\frac{\sin\dot{\varphi}^c\ \cos\dot{\theta}^c}{r^*}\frac{\partial}{\partial\theta}+\sin\dot{\varphi}^c\ \sin\dot{\theta}^c\frac{\partial}{\partial r^*},\tag{96}$$

$$\frac{\partial}{\partial \tilde{x}_3} = -\frac{\sin\dot{\theta}^c}{r^*}\frac{\partial}{\partial\theta} + \cos\dot{\theta}^c\frac{\partial}{\partial r^*}. \tag{97}$$

$$\frac{\partial}{\partial \tilde{x}_1^*} = -i\frac{\partial}{\partial \tilde{x}_1} = \frac{\sin\dot{\varphi}^c}{r^*\ \sin\dot{\theta}^c}\frac{\partial}{\partial\epsilon} - \frac{\cos\dot{\varphi}^c\ \cos\dot{\theta}^c}{r^*}\frac{\partial}{\partial\tau} - i\ \cos\dot{\varphi}^c\ \sin\dot{\theta}^c\frac{\partial}{\partial r^*}, \tag{98}$$

$$\frac{\partial}{\partial \tilde{x}_2^*} = -i\frac{\partial}{\partial \tilde{x}_2} = -\frac{\cos\dot{\varphi}^c}{r^*\ \sin\dot{\theta}^c}\frac{\partial}{\partial\epsilon} - \frac{\sin\dot{\varphi}^c\ \cos\dot{\theta}^c}{r^*}\frac{\partial}{\partial\tau} - i\ \sin\dot{\varphi}^c\ \sin\dot{\theta}^c\frac{\partial}{\partial r^*}, \tag{99}$$

$$\frac{\partial}{\partial \tilde{x}_3^*} = -i\frac{\partial}{\partial \tilde{x}_3} = \frac{\sin\theta^c}{r^*}\frac{\partial}{\partial\tau} - i\ \cos\dot{\theta}^c\frac{\partial}{\partial r^*}. \tag{100}$$

Substituting the functions $\psi = T_{\mathfrak{g}}^{-1}\psi'(\dot{\psi} = \overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot{\psi}')$ and the derivatives (89)–(94), (95)–(100) into the system (61), and multiply by $T_{\mathfrak{g}} = T(\varphi^c, \theta^c, 0)(\overset{*}{T}{}_{\mathfrak{g}} = \overset{*}{T}(\dot{\varphi}^c, \dot{\theta}^c, 0))$ from the left, we obtain

$$\begin{aligned}
&T_{\mathfrak{g}}\Lambda_1\left[-\frac{\sin\varphi^c}{r\ \sin\theta^c}\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial\varphi} + i\frac{\sin\varphi^c}{r\ \sin\theta^c}\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial\epsilon} + \frac{\cos\varphi^c\ \cos\theta^c}{r}\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial\theta}\right.\\
&\left.-\ i\frac{\cos\varphi^c\ \cos\theta^c}{r}\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial\tau} + 2\ \cos\varphi^c\ \sin\theta^c\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial r}\right]\\
&+ T_{\mathfrak{g}}\Lambda_2\left[\frac{\cos\varphi^c}{r\ \sin\theta^c}\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial\varphi} - i\frac{\cos\varphi^c}{r\ \sin\theta^c}\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial\epsilon} + \frac{\sin\varphi^c\ \cos\theta^c}{r}\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial\theta}\right.\\
&\left.-\ i\frac{\sin\varphi^c\ \cos\theta^c}{r}\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial\tau} + 2\ \sin\varphi^c\ \sin\theta^c\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial r}\right] + T_{\mathfrak{g}}\Lambda_3\\
&\times\left[-\frac{\sin\theta^c}{r}\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial\theta} + i\frac{\sin\theta^c}{r}\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial\tau} + 2\ \cos\theta^c\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial r}\right] + \kappa^c\psi' = 0,
\end{aligned}$$

$$\begin{aligned}
&\overset{*}{T}{}_{\mathfrak{g}}\Lambda_1^*\left[-\frac{\sin\dot{\varphi}^c}{r^*\ \sin\theta^c}\frac{\partial(\overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot{\psi}')}{\partial\dot{\varphi}} + i\frac{\sin\dot{\varphi}^c}{r\ \sin\dot{\theta}^c}\frac{\partial(\overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot{\psi}')}{\partial\epsilon} + \frac{\cos\dot{\varphi}^c\ \cos\dot{\theta}^c}{r^*}\right.\\
&\left.\times\frac{\partial(\overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot{\psi}')}{\partial\theta} -\ i\frac{\cos\dot{\varphi}^c\ \cos\dot{\theta}^c}{r}\frac{\partial(\overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot{\psi}')}{\partial\tau} + 2\ \cos\dot{\varphi}^c\ \sin\dot{\theta}^c\frac{\partial(\overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot{\psi}')}{\partial r^*}\right]\\
&+\ \overset{*}{T}{}_{\mathfrak{g}}\Lambda_2^*\left[\frac{\cos\dot{\varphi}^c}{r^*\ \sin\dot{\theta}^c}\frac{\partial(\overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot{\psi}')}{\partial\dot{\varphi}} - i\frac{\cos\dot{\varphi}^c}{r^*\ \sin\dot{\theta}^c}\frac{\partial(\overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot{\psi}')}{\partial\epsilon} + \frac{\sin\dot{\varphi}^c\ \cos\dot{\theta}^c}{r^*}\right.
\end{aligned}$$

$$\times \frac{\partial(\overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot{\psi}')}{\partial\dot{\theta}} - i\frac{\sin\dot{\varphi}^c\ \cos\dot{\theta}^c}{r}\frac{\partial(\overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot{\psi}')}{\partial\tau} + 2\ \sin\dot{\varphi}^c\ \sin\dot{\theta}^c\frac{\partial(\overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot{\psi}')}{\partial r^*}\Bigg]$$

$$+ \overset{*}{T}{}_{\mathfrak{g}}\Lambda_3^*\left[-\frac{\sin\dot{\theta}^c}{r^*}\frac{\partial(\overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot{\psi}')}{\partial\dot{\theta}} + i\frac{\sin\dot{\theta}^c}{r^*}\frac{\partial(\overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot{\psi}')}{\partial\tau} + 2\ \cos\dot{\theta}^c\frac{\partial(\overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot{\psi}')}{\partial r^*}\right]$$

$$+ \dot{\kappa}^c\dot{\psi}' = 0, \tag{101}$$

In virtue of the invariance conditions (80) we have

$$\begin{aligned}
T_{\mathfrak{g}}[-\Lambda_1\ \sin\varphi^c + \Lambda_2\ \cos\varphi^c]T_{\mathfrak{g}}^{-1} &= \Lambda_1,\\
T_{\mathfrak{g}}[\Lambda_1\ \cos\varphi^c\ \cos\theta^c + \Lambda_2\ \sin\varphi^c\ \cos\ \theta^c - \Lambda_3\ \sin\theta^c]T_{\mathfrak{g}}^{-1} &= \Lambda_2,\\
T_{\mathfrak{g}}[2\Lambda_1\ \cos\varphi^c\ \sin\theta^c + 2\Lambda_2\ \sin\varphi^c\ \sin\ \theta^c + 2\Lambda_3\ \cos\theta^c]T_{\mathfrak{g}}^{-1} &= 2\Lambda_3,\\
T_{\mathfrak{g}}[i\Lambda_1\ \sin\varphi^c - i\Lambda_2\ \cos\varphi^c]T_{\mathfrak{g}}^{-1} &= i\Lambda_1,\\
T_{\mathfrak{g}}[-i\Lambda_1\ \cos\varphi^c\ \cos\theta^c - i\Lambda_2\ \sin\varphi^c\ \cos\theta^c + i\Lambda_3\ \sin\theta^c] &= i\Lambda_2,\\
\overset{*}{T}{}_{\mathfrak{g}}[-\Lambda_1^*\ \sin\dot{\varphi}^c + \Lambda_2^*\ \cos\dot{\varphi}^c]\overset{*}{T}{}_{\mathfrak{g}}^{-1} &= \Lambda_1^*,\\
\overset{*}{T}{}_{\mathfrak{g}}[\Lambda_1^*\ \cos\dot{\varphi}^c\ \cos\dot{\theta}^c + \Lambda_2^*\ \sin\dot{\varphi}^c\ \cos\dot{\theta}^c - \Lambda_3^*\ \sin\dot{\theta}^c]\overset{*}{T}{}_{\mathfrak{g}}^{-1} &= \Lambda_2^*,\\
\overset{*}{T}{}_{\mathfrak{g}}[2\Lambda_1^*\ \cos\dot{\varphi}^c\ \sin\dot{\theta}^c + 2\Lambda_2^*\ \sin\dot{\varphi}^c\ \sin\dot{\theta}^c - 2\Lambda_3^*\ \cos\dot{\theta}^c]\overset{*}{T}{}_{\mathfrak{g}}^{-1} &= \Lambda_3^*,\\
\overset{*}{T}{}_{\mathfrak{g}}[i\Lambda_1^*\ \sin\dot{\varphi}^c + i\Lambda_2^*\ \cos\dot{\varphi}^c]\overset{*}{T}{}_{\mathfrak{g}}^{-1} &= -i\Lambda_1^*,\\
\overset{*}{T}{}_{\mathfrak{g}}[-i\Lambda_1^*\ \cos\dot{\varphi}^c\ \cos\dot{\theta}^c - i\Lambda_2^*\ \sin\dot{\varphi}^c\ \cos\dot{\theta}^c + i\Lambda_3^*\ \sin\dot{\theta}^c]\overset{*}{T}{}_{\mathfrak{g}}^{-1} &= -i\Lambda_2^*
\end{aligned}$$

Taking into account the latter relations we can write the system (101) as follows

$$\frac{1}{r\ \sin\theta^c}\Lambda_1 T_{\mathfrak{g}}\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial\varphi} + \frac{i}{r\ \sin\theta^c}\Lambda_1 T_{\mathfrak{g}}\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial\epsilon}$$

$$-\frac{1}{r}\Lambda_2 T_{\mathfrak{g}}\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial\theta} - \frac{i}{r}\Lambda_2 T_{\mathfrak{g}}\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial\tau} + 2\Lambda_3 T_{\mathfrak{g}}\frac{\partial(T_{\mathfrak{g}}^{-1}\psi')}{\partial r} + \kappa^c\psi' = 0,$$

$$\frac{1}{r^*\ \sin\dot{\theta}^c}\Lambda_1^*\overset{*}{T}\ \frac{\partial(\overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot{\psi}')}{\partial\varphi} - \frac{i}{r^*\ \sin\dot{\theta}^c}\Lambda_1^*\overset{*}{T}{}_{\mathfrak{g}}\frac{\partial(\overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot{\psi}')}{\partial\epsilon}$$

$$-\frac{1}{r^*}\Lambda_2^*\overset{*}{T}_{\mathfrak{g}}\frac{\partial(\overset{*}{T}_{\mathfrak{g}}\dot{\psi}')}{\partial\theta}+\frac{i}{r^*}\Lambda_2^*\overset{*}{T}_{\mathfrak{g}}\frac{\partial(\overset{*}{T}_{\mathfrak{g}}\dot{\psi}')}{\partial\tau}+2\Lambda_3^*\overset{*}{T}_{\mathfrak{g}}\frac{\partial(\overset{*}{T}_{\mathfrak{g}}\dot{\psi}')}{\partial r^*}+\dot{\kappa}^c\dot{\psi}'=0. \tag{102}$$

The matrices $T_{\mathfrak{g}}^{-1}$, $\overset{*}{T}{}_{\mathfrak{g}}^{-1}$ depend on $\varphi,\epsilon,\theta,\tau$. By this reason we must differentiate in $T_{\mathfrak{g}}^{-1}\psi'$, $\overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot{\psi}'$ the both factors. After differentiation, the system (102) takes a form

$$\begin{aligned}
&\frac{1}{r\,\sin\theta^c}\Lambda_1\frac{\partial\psi'}{\partial\varphi}+\frac{i}{r\,\sin\theta^c}\Lambda_1\frac{\partial\psi'}{\partial\epsilon}-\frac{1}{r}\Lambda_2\frac{\partial\psi'}{\partial\theta}\\
&-\frac{i}{r}\Lambda_2\frac{\partial\psi'}{\partial\tau}+2\Lambda_3\frac{\partial\psi'}{\partial r}+\left[\frac{1}{r\,\sin\theta^c}\Lambda_1T_{\mathfrak{g}}\frac{\partial T_{\mathfrak{g}}^{-1}}{\partial\varphi}\right.\\
&\left.+\frac{i}{r\,\sin\theta^c}\Lambda_1T_{\mathfrak{g}}\frac{\partial T_{\mathfrak{g}}^{-1}}{\partial\epsilon}-\frac{1}{r}\Lambda_2T_{\mathfrak{g}}\frac{\partial T_{\mathfrak{g}}^{-1}}{\partial\theta}-\frac{i}{r}\Lambda_2T_{\mathfrak{g}}\frac{\partial T_{\mathfrak{g}}^{-1}}{\partial\tau}+\kappa^cI\right]\psi'=0,\\
&\frac{1}{r^*\,\sin\dot{\theta}^c}\Lambda_1^*\frac{\partial\dot{\psi}'}{\partial\varphi}-\frac{i}{r^*\,\sin\dot{\theta}^c}\Lambda_1^*\frac{\partial\dot{\psi}'}{\partial\epsilon}-\frac{1}{r^*}\Lambda_2^*\frac{\partial\dot{\psi}'}{\partial\theta}+\frac{i}{r^*}\Lambda_2^*\frac{\partial\dot{\psi}'}{\partial\tau}\\
&+2\Lambda_3^*\frac{\partial\dot{\psi}'}{\partial r^*}+\left[\frac{1}{r^*\,\sin\dot{\theta}^c}\Lambda_1^*\overset{*}{T}_{\mathfrak{g}}\frac{\partial\overset{*}{T}{}_{\mathfrak{g}}^{-1}}{\partial\varphi}-\frac{i}{r^*\,\sin\dot{\theta}^c}\Lambda_1\overset{*}{T}_{\mathfrak{g}}\frac{\partial\overset{*}{T}{}_{\mathfrak{g}}^{-1}}{\partial\epsilon}\right.\\
&\left.-\frac{1}{r^*}\Lambda_2^*\overset{*}{T}_{\mathfrak{g}}\frac{\partial\overset{*}{T}{}_{\mathfrak{g}}^{-1}}{\partial\theta}-\frac{i}{r^*}\Lambda_2^*\overset{*}{T}_{\mathfrak{g}}\frac{\partial\overset{*}{T}{}_{\mathfrak{g}}^{-1}}{\partial\tau}+\dot{\kappa}^cI\right]\dot{\psi}'=0,
\end{aligned} \tag{103}$$

Let us show that the products $T_{\mathfrak{g}}\frac{\partial T_{\mathfrak{g}}^{-1}}{\partial\varphi},\cdots,\overset{*}{T}\frac{\partial\overset{*}{T}{}_{\mathfrak{g}}^{-1}}{\partial\tau}$ are expressed via linear combinations of the operators A_i, B_i, $\tilde{\mathsf{A}}_i$, $\tilde{\mathsf{B}}_i$. Indeed, a matrix of the fundamental representation $\mathfrak{g}(\varphi,\epsilon,\theta,\tau,0,0)=T_{\mathfrak{g}}(\varphi^c,\theta^c,0)$ of the group $\mathfrak{G}_+$ has a form (see (26))

$$\mathfrak{g}^T(\varphi,\epsilon,\theta,\tau,0,0)=\begin{pmatrix}\cos\frac{\theta^c}{2}\,e^{\frac{i\varphi^c}{2}} & i\,\sin\frac{\theta^c}{2}\,e^{-\frac{i\varphi^c}{2}}\\ i\,\sin\frac{\theta^c}{2}\,e^{\frac{i\varphi^c}{2}} & \cos\frac{\theta^c}{2}\,e^{-\frac{i\varphi}{2}}\end{pmatrix}=$$

$$\begin{pmatrix}e^{\frac{\epsilon+i\varphi}{2}}\left[\cos\frac{\theta}{2}\,\cosh\frac{\tau}{2}+i\,\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\right] & e^{\frac{-\epsilon-i\varphi}{2}}\left[\cos\frac{\theta}{2}\,\sinh\frac{\tau}{2}+i\,\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\right]\\ e^{\frac{\epsilon+i\varphi}{2}}\left[\cos\frac{\theta}{2}\,\sinh\frac{\tau}{2}+i\,\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\right] & e^{\frac{-\epsilon-i\varphi}{2}}\left[\cos\frac{\tau}{2}\,\sinh\frac{\tau}{2}+i\,\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\right]\end{pmatrix}$$

and its inverse matrix is

$$(\mathfrak{g}^T)^{-1} = \begin{pmatrix} e^{\frac{-\epsilon-i\varphi}{2}}\left[\cos\frac{\theta}{2}\ \cosh\frac{\tau}{2}+i\ \sin\frac{\theta}{2}\sinh\frac{\tau}{2}\right] & -e^{\frac{-\epsilon-i\varphi}{2}}\left[\cos\frac{\theta}{2}\ \sinh\frac{\tau}{2}+i\ \sin\frac{\theta}{2}\cosh\frac{\tau}{2}\right] \\ -e^{\frac{\epsilon+i\varphi}{2}}\left[\cos\frac{\theta}{2}\ \sinh\frac{\tau}{2}+i\ \sin\frac{\theta}{2}\cosh\frac{\tau}{2}\right] & e^{\frac{\epsilon-i\varphi}{2}}\left[\cos\frac{\tau}{2}\ \cosh\frac{\theta}{2}+i\ \sin\frac{\theta}{2}\sinh\frac{\tau}{2}\right] \end{pmatrix}$$

In accordance with (10), (14), and (11)–(13) infinitesimal operators for the fundamental representation $\mathfrak{g}$ are

$$\mathsf{A}_1 = -\frac{i}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \mathsf{A}_2 = -\frac{1}{2}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\quad \mathsf{A}_3 = -\frac{1}{2}\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix},$$
$$\mathsf{B}_1 = -\frac{1}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \mathsf{B}_2 = -\frac{1}{2}\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix},\quad \mathsf{B}_3 = -\frac{1}{2}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} \qquad (104)$$

Then

$$\mathfrak{g}^T\frac{\partial(\mathfrak{g}^T)^{-1}}{\partial\varphi} = \frac{i}{2}\begin{pmatrix}-\cos\theta^c & i\ \sin\theta^c\\ -i\ \sin\theta^c & \cos\theta^c\end{pmatrix} = -\mathsf{A}_3\ \cos\theta^c - \mathsf{A}_2\sin\theta^c, \qquad (105)$$

$$\mathfrak{g}^T\frac{\partial(\mathfrak{g}^T)^{-1}}{\partial\epsilon} = \frac{1}{2}\begin{pmatrix}-\cos\theta^c & i\ \sin\theta^c\\ -i\ \sin\theta^c & \cos\theta^c\end{pmatrix} = -\mathsf{B}_3\ \cos\theta^c - \mathsf{B}_2\sin\theta^c, \qquad (106)$$

$$\mathfrak{g}^T\frac{\partial(\mathfrak{g}^T)^{-1}}{\partial\theta} = \frac{1}{2}\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix} = \mathsf{A}_1, \qquad (107)$$

$$\mathfrak{g}^T\frac{\partial(\mathfrak{g}^T)^{-1}}{\partial\tau} = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \mathsf{B}_1, \qquad (108)$$

Further, a matrix of the conjugate fundamental representation $\dot{\mathfrak{g}}(\varphi, \epsilon, \theta, \tau, 0, 0) = \overset{*}{T}\ \mathfrak{g}(\dot{\varphi}^c, \dot{\theta}^c, 0)$ of the group $\mathfrak{G}_+$ has a form

$$\dot{\mathfrak{g}}^T(\varphi, \epsilon, \theta, \tau, 0, 0) = \begin{pmatrix} \cos\frac{\dot{\theta}^c}{2}\ e^{-\frac{i\dot{\varphi}^c}{2}} & -i\ \sin\frac{\dot{\theta}^c}{2}\ e^{\frac{i\varphi}{2}} \\ -i\ \sin\frac{\dot{\theta}^c}{2}e^{-\frac{i\dot{\varphi}^c}{2}} & \cos\frac{\dot{\theta}^c}{2}\ e^{\frac{i\dot{\varphi}^c}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\frac{\epsilon-i\varphi}{2}}\left[\cos\frac{\theta}{2}\ \cosh\frac{\tau}{2} - i\ \sin\frac{\theta}{2}\ \sinh\frac{\tau}{2}\right] & e^{\frac{\epsilon-i\varphi}{2}}\left[\cos\frac{\theta}{2}\ \cosh\frac{\tau}{2} - i\ \sin\frac{\theta}{2}\ \sinh\frac{\tau}{2}\right] \\ e^{\frac{\epsilon-i\varphi}{2}}\left[\cos\frac{\theta}{2}\ \sin\frac{\tau}{2} - i\ \sin\frac{\theta}{2}\ \cosh\frac{\tau}{2}\right] & e^{\frac{\epsilon-i\varphi}{2}}\left[\cos\frac{\theta}{2}\ \cosh\frac{\tau}{2} - i\ \sin\frac{\theta}{2}\ \sinh\frac{\tau}{2}\right] \end{pmatrix}$$

and its inverse matrix is

$$(\dot{\mathfrak{g}}^T)^{-1} =$$
$$\begin{pmatrix} e^{\frac{\epsilon+i\varphi}{2}}\left[\cos\frac{\theta}{2}\ \cosh\frac{\tau}{2} - i\ \sin\frac{\theta}{2}\ \sinh\frac{\tau}{2}\right] & -e^{\frac{-\epsilon+i\varphi}{2}}\left[\cos\frac{\theta}{2}\ \sinh\frac{\tau}{2} - i\ \sin\frac{\theta}{2}\ \cosh\frac{\tau}{2}\right] \\ -e^{\frac{\epsilon-i\varphi}{2}}\left[\cos\frac{\theta}{2}\ \sin\frac{\tau}{2} - i\ \sin\frac{\theta}{2}\ \cosh\frac{\tau}{2}\right] & e^{\frac{\epsilon-i\varphi}{2}}\left[\cos\frac{\theta}{2}\ \cosh\frac{\tau}{2} - i\ \sin\frac{\theta}{2}\ \sinh\frac{\tau}{2}\right] \end{pmatrix}.$$

In accordance with (15) and (16) infinitesimal operators for the conjugate fundamental representation $\dot{\mathfrak{g}}$ are

$$\tilde{\mathsf{A}}_1 = -\frac{i}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tilde{\mathsf{A}}_2 = -\frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \tilde{\mathsf{A}}_3 = -\frac{1}{2}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},$$

$$\tilde{\mathsf{B}}_1 = -\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tilde{\mathsf{B}}_2 = -\frac{1}{2}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \tilde{\mathsf{B}}_3 = -\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{109}$$

In this case we have

$$\dot{\mathfrak{g}}^T \frac{\partial(\dot{\mathfrak{g}}^T)^{-1}}{\partial\varphi} = \frac{i}{2}\begin{pmatrix} \cos\dot{\theta}^c & i\,\sin\dot{\theta}^c \\ -i\,\sin\dot{\theta}^c & -\cos\dot{\theta}^c \end{pmatrix} = -\tilde{\mathsf{A}}_3\,\cos\dot{\theta}^c + \tilde{\mathsf{A}}_2\sin\dot{\theta}^c, \tag{110}$$

$$\dot{\mathfrak{g}}^T \frac{\partial(\dot{\mathfrak{g}}^T)^{-1}}{\partial\epsilon} = \frac{1}{2}\begin{pmatrix} -\cos\dot{\theta}^c & i\,\sin\dot{\theta}^c \\ -i\,\sin\dot{\theta}^c & -\cos\dot{\theta}^c \end{pmatrix} = -\tilde{\mathsf{B}}_3\,\cos\dot{\theta}^c - \tilde{\mathsf{B}}_2\sin\dot{\theta}^c, \tag{111}$$

$$\dot{\mathfrak{g}}^T \frac{\partial(\dot{\mathfrak{g}}^T)^{-1}}{\partial\dot{\theta}} = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \tilde{\mathsf{A}}_1, \tag{112}$$

$$\dot{\mathfrak{g}}^T \frac{\partial(\mathfrak{g}^T)^{-1}}{\partial\tau} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \tilde{\mathsf{B}}_1, \tag{113}$$

It is easy to verify that relations (105)–(113) take place for any representation $\mathfrak{g} \rightarrow T_{\mathfrak{g}}$ of the group $\mathfrak{G}_+$. Substituting these relations into the system (103) we obtain

$$\begin{aligned}
&\frac{1}{r\,\sin\theta^c}\Lambda_1\frac{\partial\psi'}{\partial\varphi} + \frac{i}{r\,\sin\theta^c}\Lambda_1\frac{\partial\psi'}{\partial\epsilon} - \frac{1}{r}\Lambda_2\frac{\partial\psi'}{\partial\theta} - \frac{i}{r}\Lambda_2\frac{\partial\psi'}{\partial\tau} \\
&+2\Lambda_3\frac{\partial\dot{\psi}'}{\partial r} - \frac{1}{r}[-(\Lambda_1\mathsf{A}_2 + \Lambda_2\mathsf{A}_1) + i(\Lambda_1\mathsf{B}_2 - \Lambda_2\mathsf{B}_1) \\
&+\cot\theta^c(i\Lambda_1\mathsf{B}_3 - \Lambda_1\mathsf{A}_3)]\psi' + \kappa^c\psi = 0, \\
&\frac{1}{r^*\,\sin\dot{\theta}^c}\Lambda_1^*\frac{\partial\dot{\psi}'}{\partial\varphi} - \frac{i}{r^*\,\sin\dot{\theta}^c}\Lambda_1^*\frac{\partial\dot{\psi}'}{\partial\epsilon} + \frac{1}{r^*}\Lambda_2^*\frac{\partial\dot{\psi}'}{\partial\dot{\theta}} - \frac{i}{r^*}\Lambda_2^*\frac{\partial\dot{\psi}'}{\partial\tau} \\
&+2\Lambda_3^*\frac{\partial\dot{\psi}'}{\partial r^*} + \frac{1}{r^*}\big[\big(\Lambda_1^*\tilde{\mathsf{A}}_2 + \Lambda_2^*\tilde{\mathsf{A}}_1\big) + i\big(\Lambda_1^*\tilde{\mathsf{B}}_2 - \Lambda_2^*\tilde{\mathsf{B}}_1\big) \\
&+\cot\dot{\theta}^c\big(i\Lambda_1^*\tilde{\mathsf{B}}_3 - \Lambda_1^*\tilde{\mathsf{A}}_3\big)\big]\dot{\psi}' + \dot{\kappa}^c\dot{\psi} = 0.
\end{aligned} \tag{114}$$

Now we are in a position that allows to separate variables in relativistically invariant system. We decompose the each component ψ^k_{lm} of the wave function ψ into the series on the generalized hyperspherical functions. This procedure gives rise to separation of variables, that is, it reduces the relativistically invariant system to a system of ordinary differential equations. Preliminarily, let us calculate elements of the matrices $V = \Lambda_1 \mathsf{B}_2 - \Lambda_2 \mathsf{B}_1$, $U = \Lambda_1^* \tilde{\mathsf{A}}_2 - \Lambda_2^* \tilde{\mathsf{A}}_1$, $G = \Lambda_1 \mathsf{A}_2 + \Lambda_2 \mathsf{A}_1$, $W = \Lambda_1^* \tilde{\mathsf{B}}_2 + \Lambda_2^* \tilde{\mathsf{B}}_1$.

First of all, let us find elements of the matrix $V = \Lambda_1 \mathsf{B}_2 - \Lambda 2 \mathsf{B}_1$. Using the relations (69) we can write $V = 2i\Lambda_3 + \mathsf{B}_2\Lambda_1 - \mathsf{B}_1\Lambda_2$. Let

$$V\xi^k_{l,m} = \sum_{l',m',k'} v^{k'k}_{l'l,m'm}\xi^{k'}_{l',m'}$$

be an expression of the matrix V for the canonical basis $\{\xi^k_{lm}\}$. Taking into account (62)–(64) and (14) we obtain

$$\begin{aligned}
V\xi^k_m &= (2i\Lambda_3 + \mathsf{B}_2\Lambda_1 - \mathsf{B}_1\Lambda_2)\xi^k_{l,m} \\
&= 2i\sum_{l',m',k'} c^{k'k}_{l'l,m'm}\xi^{k'}_{l',m'} + \mathsf{B}_2\sum_{l',m',k'} a^{k'k}_{l'l,m'm}\xi^{k'}_{l',m'} - \mathsf{B}_1\sum_{l',m',k'} b^{k'k}_{l'l,m'm}\xi^{k'}_{l',m'} \\
&= 2i\sum_{l',m',k'} c^{k'k}_{l'l,m'm}\xi^{k'}_{l',m'} + \frac{i}{2}\sum_{l',m',k'} a^{k'k}_{l'l,m'm}\left(\alpha^{l'}_{m'}\xi^{k'}_{l',m'-1} - \alpha^{l'}_{m'+1}\xi^{k'}_{l',m'+1}\right) \\
&\quad - \frac{1}{2}\sum_{l',m',k'} b^{k'k}_{l'l,m'm}\left(\alpha^{l'}_{m}\xi^{k'}_{l',m'-1} + \alpha^{l'}_{m'+1}\xi^{k'}_{l',m'+1}\right).
\end{aligned}$$

Dividing each of the two latter sums on the two and changing the summation index in the each four obtained sums, we come to the following expression

$$\begin{aligned}
V\xi^k_{l,m} = \sum_{l',m',k'}\sum_{l',m',k'} &\left(2ic^{k'k}_{l'l,m'm} - \frac{i}{2}\alpha^{l'}_{m'}a^{l'}_{l'l,m'-1,m}\right. \\
&\left. + \frac{i}{2}\alpha^{l'}_{m'+1}a^{k'k}_{l'l,m'+1,m} - \frac{1}{2}\alpha^{l'}_{m'}b^{k'k}_{l'l,m'-1,m} - \frac{1}{2}\alpha^{l'}_{m'+1}b^{k'k}_{l'l,m'+1,m}\right)\xi^{k'}_{l',m'}.
\end{aligned}$$

Therefore, a general element of the matrix V have a form

$$\begin{aligned}
v^{k'k}_{l'l,m'm} = 2ic^{k'k}_{l'l,m'm} &- \frac{i}{2}\alpha^{l'}_{m'}a^{k'k}_{l'l,m'-1,m} + \frac{i}{2}\alpha^{l'}_{m'+1}a^{k'k}_{l'l,m'+1,m} \\
&- \frac{1}{2}\alpha^{l'}_{m'}b^{k'k}_{l'l,m'-1,m} - \frac{1}{2}\alpha^{l'}_{m'+1}b^{k'k}_{l'l,m'+1,m}.
\end{aligned}$$

Using the formulae (62)–(64), we find that

$$V:\begin{cases}
v^{k'k}_{l-1,l,mm} &= ic^{k'k}_{l-1,l}(l+1)\sqrt{l^2-m^2}, \\
v^{k'k}_{ll,mm} &= ic^{k'k}_{ll}m, \\
v^{k'k}_{l+1,l,mm} &= -ic^{k'k}_{l+1,l}l\sqrt{(l+1)^2-m^2}.
\end{cases} \qquad (115)$$

All other elements $v^{k'k}_{l'l,m'm}$ are equal to zero.

Analogously, using the relations (70) and the operators (15) we find that elements of the matrix $U = \Lambda_1^* \tilde{\mathsf{A}}_2 - \Lambda_2^* \tilde{\mathsf{A}}_1$ are

$$U : \begin{cases} u^{\dot{k}'\dot{k}}_{\dot{l}-1,\dot{l},\dot{m}\dot{m}} & = & c^{\dot{k}'\dot{k}}_{\dot{l}-1,\dot{l}}(\dot{l}+1)\sqrt{\dot{l}^2-\dot{m}^2}, \\ u^{\dot{k}'\dot{k}}_{\dot{l}\dot{l},\dot{m}\dot{m}} & = & -c^{\dot{k}'\dot{k}}_{\dot{l}\dot{l}}\dot{m}, \\ u^{\dot{k}'\dot{k}}_{\dot{l}+1,\dot{l},\dot{m}\dot{m}} & = & c^{\dot{k}'\dot{k}}_{\dot{l}+1,\dot{l}}\dot{l}\sqrt{(\dot{l}+1)^2-\dot{m}^2}. \end{cases} \tag{116}$$

Further, using (68), (10), (71), and (16) it is easy to verify that all the elements of the matrices $G = \Lambda_1 \mathsf{A}_1 + \Lambda_2 \mathsf{A}_1$ and $W = \Lambda_1^* \tilde{\mathsf{B}}_2 + \Lambda_2^* \tilde{\mathsf{B}}_2$ are equal to zero.

The system (114) in the components ψ^k_{lm} is written as follows

$$\frac{1}{r\,\sin\theta^c}\sum_{l',m',k'} a^{kk'}_{ll',mm'}\frac{\partial\psi^{k'}_{l'm'}}{\partial\varphi} + \frac{i}{r\,\sin\theta^c}\sum_{l',m',k'} a^{kk'}_{ll',mm'}\frac{\partial\psi^{k'}_{l'm'}}{\partial\epsilon}$$

$$-\frac{1}{r}\sum_{l',m',k'} b^{kk'}_{ll',mm'}\frac{\partial\psi^{k'}_{l'm'}}{\partial\theta} - \frac{i}{r}\sum_{l',m',k'} b^{kk'}_{ll',mm'}\frac{\partial\psi^{k'}_{l'm'}}{\partial\tau} + 2\sum_{l',m',k'} c^{kk'}_{ll',mm'}\frac{\partial\psi^{k'}_{l'm'}}{\partial r}$$

$$+\frac{i}{r}\sum_{l',m',k'} v^{kk'}_{ll',mm'}\psi^{k'}_{l'm'} + \frac{2i}{r}\cot\theta^c\sum_{l',m',k'} m' a^{kk'}_{ll',mm'}\psi^{k'}_{l'm'} + \kappa^c\psi^k_{lm} = 0,$$

$$\frac{1}{r^*\,\sin\dot{\theta}^c}\sum_{\dot{l}',\dot{m}',\dot{k}'} a^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}',m\dot{m}'}\frac{\partial\psi^{\dot{k}'}_{\dot{l}'\dot{m}'}}{\partial\varphi} + \frac{i}{r^*\,\sin\dot{\theta}^c}\sum_{\dot{l}',\dot{m}',\dot{k}'} d^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}',m\dot{m}'}\frac{\partial\psi^{\dot{k}'}_{\dot{l}'\dot{m}'}}{\partial\epsilon}$$

$$-\frac{1}{r^*}\sum_{\dot{l}',\dot{m}',\dot{k}'} b^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}',m\dot{m}'}\frac{\partial\psi^{\dot{k}'}_{\dot{l}'\dot{m}'}}{\partial\theta} - \frac{i}{r^*}\sum_{\dot{l}',\dot{m}',\dot{k}'} b^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}',\dot{m}\dot{m}'}\frac{\partial\psi^{\dot{k}'}_{\dot{l}'\dot{m}'}}{\partial\tau} + 2\sum_{\dot{l}',\dot{m}',\dot{k}'} f^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}',m\dot{m}'}\frac{\partial\psi^{\dot{k}'}_{\dot{l}'\dot{m}'}}{\partial r^*}$$

$$+\frac{i}{r^*}\sum_{\dot{l}',\dot{m}',\dot{k}'} v^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}',m\dot{m}'}\psi^{\dot{k}'}_{\dot{l}'\dot{m}'} + \frac{2i}{r}\cot\theta^c\sum_{\dot{l}',\dot{m}',\dot{k}'} \dot{m}' d^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}',m\dot{m}'}\psi^{\dot{k}'}_{\dot{l}'\dot{m}'} + \dot{\kappa}^c\psi^{\dot{k}}_{\dot{l}m} = 0, \tag{117}$$

where the coefficients $a^{k'k}ll', mm'$, $b^{kk'}_{ll',mm'}$, $c^{kk'}_{ll',mm'}$, $v^{kk'}_{ll',mm'}$, $d^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}',\dot{m}\dot{m}'}$, $e^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}',\dot{m}\dot{m}'}$ $f^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}',\dot{m}\dot{m}'}$ $u^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}',\dot{m}\dot{m}'}$ are defined by the formulae (62)–(64), (115), (65)–(67), (116).

With a view to separate the variables in (117) let us assume

$$\psi^k_{lm} = f^{l_0}_{lmk}(r)\,\mathfrak{M}^{l_0}_{mn}(\varphi,\epsilon,\theta,\tau,0,0), \tag{118}$$

where $l_0 \geq l$, and $-l_0 \leq m, n \leq l_0$. Substituting the functions (118) into the system (117) and taking into account values of the coefficients $a^{kk'}_{ll',mm'}$, $b^{kk'}_{ll',mm'}$, $c^{kk'}_{ll',mm'}$, $v'^{kk}_{ll',mm'}$ $d^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}',\dot{m}\dot{m}'}$, $e^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}',\dot{m}\dot{m}'}$, $f^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}',\dot{m}\dot{m}'}$, $u^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}',\dot{m}\dot{m}'}$ let us collect together the terms with identical radical functions. In the result we obtain

$$\sum_{k'} c^{kk'}_{l,l-1}\left\{\left[2\sqrt{l^2-m^2}\frac{\partial \boldsymbol{f}^{l_0}_{l-1,m,k'}}{\partial r} - \frac{1}{r}(l+1)\sqrt{l^2-m^2}\,\boldsymbol{f}^{l_0}_{l-1,m,k'}\right]\right.$$

$$\times\, \mathfrak{M}^{l_0}_{mn}(\varphi, \epsilon, \theta, \tau, 0, 0) + \frac{1}{2r}\sqrt{(l+m)(l+m-1)}\,\boldsymbol{f}^{l_0}_{l-1,m-1,k'}$$

$$\times \left[-\frac{1}{\sin\theta^c}\frac{\partial \mathfrak{M}^{l_0}_{m-1,n}}{\partial\varphi} - \frac{i}{\sin\theta^c}\frac{\partial \mathfrak{M}^{l_0}_{m_1,n}}{\partial\epsilon} + i\frac{\partial \mathfrak{M}^{l_0}_{m-1,n}}{\partial\theta} - \frac{\partial \mathfrak{M}^{l_0}_{m-1,n}}{\partial\tau}\right.$$

$$\left. -\ \frac{2i(m-1)\cos\theta^c}{\sin\theta^c}\mathfrak{M}^{l_0}_{m-1,n}\right] + \frac{1}{2r}\sqrt{(l-m)(l-m-1)}\,\boldsymbol{f}^{l_0}_{l-1,m+1,k'}$$

$$\times \left[\frac{1}{\sin\theta^c}\frac{\partial \mathfrak{M}^{l_0}_{m+1,n}}{\partial\varphi} + \frac{i}{\sin\theta^c}\frac{\partial \mathfrak{M}^{l_0}_{m+1,n}}{\partial\epsilon} + i\frac{\partial \mathfrak{M}^{l_0}_{m+1,n}}{\partial\theta} - \frac{\partial \mathfrak{M}^{l_0}_{m+1,n}}{\partial\tau}\right.$$

$$\left. +\ \frac{2i(m+1)\cos\theta^c}{\sin\theta^c}\mathfrak{M}^{l_0}_{m+1,n}\right]\Bigg\} + \sum_{k'} c^{kk'}_{ll}\left\{\left[2m\frac{\partial\, \boldsymbol{f}^{l^0}_{l,m,k'}}{\partial r} - \frac{1}{r}m\,\boldsymbol{f}^{l_0}_{l,m,k'}\right]\right.$$

$$\times\, \mathfrak{M}^{l_0}_{m,n}(\varphi, \epsilon, \theta, \tau, 0, 0) + \frac{1}{2r}\sqrt{(l+m)(l-m+1)}\,\boldsymbol{f}^{l_0}_{l,m-1,k'}$$

$$\times \left[\frac{1}{\sin\theta^c}\frac{\partial \mathfrak{M}^{l_0}_{m-1,n}}{\partial\varphi} + \frac{i}{\sin\theta^c}\frac{\partial \mathfrak{M}^{l_0}_{m-1,n}}{\partial\epsilon} - i\frac{\partial \mathfrak{M}^{l_0}_{m-1,n}}{\partial\theta} + \frac{\partial \mathfrak{M}^{l_0}_{m-1,n}}{\partial\tau}\right.$$

$$\left. +\ \frac{2i(m-1)\cos\theta^c}{\sin\theta^c}\mathfrak{M}^{l_0}_{m-1,n}\right] + \frac{1}{2r}\sqrt{(l+m+1)(l-m)}\,\boldsymbol{f}^{l_0}_{l,m+1,k'}$$

$$\times \left[\frac{1}{\sin\theta^c}\frac{\partial \mathfrak{M}^{l_0}_{m+1,n}}{\partial\varphi} + \frac{i}{\sin\theta^c}\frac{\partial \mathfrak{M}^{l_0}_{n+1,n}}{\partial\epsilon} + i\frac{\partial \mathfrak{M}^{l_0}_{m+1,n}}{\partial\theta} - \frac{\partial \mathfrak{M}^{l_0}_{m+1,n}}{\partial\tau}\right.$$

$$\left. +\ \frac{2i(m+1)\cos\theta^c}{\sin\theta^c}\mathfrak{M}^{l_0}_{m+1,n}\right]\Bigg\} + \sum_{k'} c^{kk'}_{l,l+1}\left\{\left[2\sqrt{(l+1)^2-m^2}\frac{\partial\, \boldsymbol{f}^{l_0}_{l+1,m,k'}}{\partial r}\right.\right.$$

$$\left. +\ \frac{1}{r}l\sqrt{(l+1)^2-m^2}\,\boldsymbol{f}^{l_0}_{l+1,m,k'}\right]\mathfrak{M}^{l_0}_{m,n}(\varphi, \epsilon, \theta, \tau, 0, 0)$$

$$+ \frac{1}{2r}\sqrt{(l-m+1)(l-m+2)}\,\boldsymbol{f}^{l_0}_{l+1,m-1,k'}\left[\frac{1}{\sin\theta^c}\frac{\partial \mathfrak{M}^{l_0}_{m-1,n}}{\partial\varphi} + \frac{i}{\sin\theta^c}\right.$$

$$\left. \times\ \frac{\partial \mathfrak{M}^{l_0}_{m-1,n}}{\partial\epsilon} - i\frac{\partial \mathfrak{M}^{l_0}_{m-1,n}}{\partial\theta} + \frac{\partial \mathfrak{M}^{l_0}_{m-1,n}}{\partial\tau} + \frac{2i(m-1)\cos\theta^c}{\sin\theta^c}\mathfrak{M}^{l_0}_{m-1,n}\right]$$

$$+ \frac{1}{2r}\sqrt{(l+m+1)(l+m+2)}\,\boldsymbol{f}^{l_0}_{l+1,m+1,k'}\left[-\frac{1}{\sin\theta^c}\frac{\partial \mathfrak{M}^{l_0}_{m+1,n}}{\partial\varphi} - \frac{i}{\sin\theta^c}\right.$$

$$\times \frac{\partial \mathfrak{M}^{l_0}_{m+1,n}}{\partial \epsilon} - i\frac{\partial \mathfrak{M}^{l_0}_{m+1,n}}{\partial \theta} + \frac{\partial \mathfrak{M}^{l_0}_{m+1,n}}{\partial \tau} - \frac{2i(m+1)\cos\theta^c}{\sin\theta^c}\mathfrak{M}^{l_0}_{m+1,n}\Bigg]\Bigg\}$$

$$+\kappa^c \boldsymbol{f}^{l_0}_{lm}\mathfrak{M}^{l_0}_{m,n}(\varphi,\epsilon,\theta,\tau,0,0) = 0, \sum_{\dot{k}'} c^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}-1}\Bigg\{\Bigg[2\sqrt{\dot{l}^2-\dot{m}^2}\frac{\partial \boldsymbol{f}^{\dot{l}_0}_{\dot{l}-1,\dot{m},\dot{k}'}}{\partial r^*}$$

$$-\frac{1}{r^*}(\dot{l}+1)\sqrt{\dot{l}^2-\dot{m}^2}\,\boldsymbol{f}^{\dot{l}_0}_{\dot{l}-1,\dot{m},\dot{k}'}\Bigg]\mathfrak{M}^{\dot{l}_0}_{\dot{m},\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0)$$

$$+\frac{1}{2r^*}\sqrt{(\dot{l}+\dot{m})(\dot{l}+\dot{m}-1)}\,\boldsymbol{f}^{\dot{l}_0}_{\dot{l}-1,\dot{m}-1,\dot{k}'}\Bigg[-\frac{1}{\sin\dot{\theta}^c}\frac{\partial \mathfrak{M}^{\dot{l}_0}_{\dot{m}-1,\dot{n}}}{\partial \varphi} + \frac{i}{\sin\dot{\theta}^c}$$

$$\times \frac{\partial \mathfrak{M}^{\dot{l}_0}_{\dot{m}-1,\dot{n}}}{\partial \epsilon} + i\frac{\partial \mathfrak{M}^{\dot{l}_0}_{\dot{m}-1,\dot{n}}}{\partial \theta} + \frac{\partial \mathfrak{M}^{\dot{l}_0}_{\dot{m}-1,\dot{n}}}{\partial \tau} + \frac{2i(\dot{m}-1)\cos\dot{\theta}^c}{\sin\dot{\theta}^c}\mathfrak{M}^{\dot{l}_0}_{\dot{m}-1,\dot{n}}\Bigg]$$

$$+\frac{1}{2r^*}\sqrt{(\dot{l}-\dot{m})(\dot{l}-\dot{m}-1)}\,\boldsymbol{f}^{\dot{l}_0}_{\dot{l}-1,\dot{m}+1,\dot{k}'}\Bigg[\frac{1}{\sin\dot{\theta}^c}\frac{\partial \mathfrak{M}^{\dot{l}_0}_{\dot{m}+1,\dot{n}}}{\partial \varphi} - \frac{i}{\sin\dot{\theta}^c}\frac{\partial \mathfrak{M}^{\dot{l}_0}_{\dot{m}+1,\dot{n}}}{\partial \epsilon}$$

$$+\ i\frac{\partial \mathfrak{M}^{\dot{l}_0}_{\dot{m}+1,\dot{n}}}{\partial \theta} + \frac{\partial \mathfrak{M}^{\dot{l}_0}_{\dot{m}+1,\dot{n}}}{\partial \tau} - \frac{2i(\dot{m}+1)\cos\dot{\theta}^c}{\sin\dot{\theta}^c}\mathfrak{M}^{\dot{l}_0}_{\dot{m}+1,\dot{n}}\Bigg]\Bigg\} + \sum_{\dot{k}'} c^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}}$$

$$\times\Bigg\{\Bigg[2\dot{m}\frac{\partial \boldsymbol{f}^{\dot{l}_0}_{\dot{l},\dot{m},\dot{k}'}}{\partial r^*} - \frac{1}{r^*}\dot{m}\,\boldsymbol{f}^{\dot{l}_0}_{\dot{l},\dot{m},\dot{k}'}\Bigg]\mathfrak{M}^{\dot{l}_0}_{\dot{m},\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0)$$

$$+\frac{1}{2r^*}\sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m}+1)}\,\boldsymbol{f}^{\dot{l}_0}_{\dot{l},\dot{m}-1,\dot{k}'}\Bigg[\frac{1}{\sin\dot{\theta}^c}\frac{\partial \mathfrak{M}^{\dot{l}_0}_{\dot{m}-1,\dot{n}}}{\partial \varphi} - \frac{i}{\sin\dot{\theta}^c}$$

$$\times \frac{\partial \mathfrak{M}^{\dot{l}_0}_{\dot{m}-1,\dot{n}}}{\partial \epsilon} - i\frac{\partial \mathfrak{M}^{\dot{l}_0}_{\dot{m}-1,\dot{n}}}{\partial \theta} - \frac{\partial \mathfrak{M}^{\dot{l}_0}_{\dot{m}-1,\dot{n}}}{\partial \tau} - \frac{2i(\dot{m}-1)\cos\dot{\theta}^c}{\sin\dot{\theta}^c}\mathfrak{M}^{\dot{l}_0}_{\dot{m}-1,\dot{n}}\Bigg]$$

$$+\frac{1}{2r^*}\sqrt{(\dot{l}+\dot{m}+1)(\dot{l}-\dot{m})}\,\boldsymbol{f}^{\dot{l}_0}_{\dot{l},\dot{m}+1,\dot{k}'}\Bigg[\frac{1}{\sin\dot{\theta}^c}\frac{\partial \mathfrak{M}^{\dot{l}_0}_{\dot{m}+1,\dot{n}}}{\partial \varphi} - \frac{i}{\sin\dot{\theta}^c}$$

$$\times \frac{\partial \mathfrak{M}^{\dot{l}_0}_{\dot{m}+1,\dot{n}}}{\partial \epsilon} + i\frac{\partial \mathfrak{M}^{\dot{l}_0}_{\dot{m}+1,\dot{n}}}{\partial \theta} + \frac{\partial \mathfrak{M}^{\dot{l}_0}_{\dot{m}+1,\dot{n}}}{\partial \tau} - \frac{2i(\dot{m}+1)\cos\dot{\theta}^c}{\sin\dot{\theta}^c}\mathfrak{M}^{\dot{l}_0}_{\dot{m}+1,\dot{n}}\Bigg]\Bigg\}$$

$$+\sum_{\dot{k}'} c^{\dot{k}\dot{k}'}_{\dot{l},\dot{l}+1}\Bigg\{\Bigg[2\sqrt{(\dot{l}+1)^2-\dot{m}^2}\frac{\partial \boldsymbol{f}^{\dot{l}_0}_{\dot{l}+1,\dot{m},\dot{k}'}}{\partial r^*} + \frac{1}{r^*}\sqrt{(\dot{l}+1)^2-\dot{m}^2}\,\boldsymbol{f}^{\dot{l}_0}_{\dot{l}+1,\dot{m},\dot{k}'}\Bigg]$$

$$+\mathfrak{M}^{\dot{l}_0}_{\dot{m},\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0)+\frac{1}{2r^*}\sqrt{(\dot{l}-\dot{m}+1)(\dot{l}-\dot{m}+2)}\,\boldsymbol{f}^{\dot{l}_0}_{\dot{l}+1,\dot{m}-1,\dot{k}'}$$

$$\times\left[\frac{1}{\sin\dot{\theta}^c}\frac{\partial\mathfrak{M}^{\dot{l}_0}_{\dot{m}-1,\dot{n}}}{\partial\varphi}-\frac{i}{\sin\dot{\theta}^c}\frac{\partial\mathfrak{M}^{\dot{l}_0}_{\dot{m}-1,\dot{n}}}{\partial\epsilon}-i\frac{\partial\mathfrak{M}^{\dot{l}_0}_{\dot{m}-1,\dot{n}}}{\partial\theta}-\frac{\partial\mathfrak{M}^{\dot{l}_0}_{\dot{m}-1,\dot{n}}}{\partial\tau}\right.$$

$$\left.-\frac{2i(\dot{m}-1)\cos\dot{\theta}^c}{\sin\dot{\theta}^c}\mathfrak{M}^{\dot{l}_0}_{\dot{m}-1,\dot{n}}\right]+\frac{1}{2r^*}\sqrt{(\dot{l}+\dot{m}+1)(\dot{l}+\dot{m}+2)}\,\boldsymbol{f}^{\dot{l}_0}_{\dot{l}+1,\dot{m}+1,\dot{k}'}$$

$$\times\left[-\frac{1}{\sin\dot{\theta}^c}\frac{\partial\mathfrak{M}^{\dot{l}_0}_{\dot{m}+1,\dot{n}}}{\partial\varphi}+\frac{i}{\sin\dot{\theta}^c}\frac{\partial\mathfrak{M}^{\dot{l}_0}_{\dot{m}+1,\dot{n}}}{\partial\epsilon}-i\frac{\partial\mathfrak{M}^{\dot{l}_0}_{\dot{m}+1,\dot{n}}}{\partial\theta}-\frac{\partial\mathfrak{M}^{\dot{l}_0}_{\dot{m}+1,\dot{n}}}{\partial\tau}\right.$$

$$\left.\left.+\frac{2i(\dot{m}+1)\cos\dot{\theta}^c}{\sin\dot{\theta}^c}\mathfrak{M}^{\dot{l}_0}_{\dot{m}+1,\dot{n}}\right]\right\}+\dot{\kappa}^c\,\boldsymbol{f}^{\dot{l}_0}_{\dot{l}\dot{m}}\mathfrak{M}^{\dot{l}_0}_{\dot{m}\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0)=0. \qquad (119)$$

Each equation of the obtained system contains three generalized hyperspherical functions $\mathfrak{M}^{l_0}_{mn}$, $\mathfrak{M}^{l_0}_{m-1,n}$, $\mathfrak{M}^{l_0}_{m+1,n}$ and their conjugate. We apply the recurrence relations (56)–(59) to square brackets containing the functions $\mathfrak{M}^{l_0}_{m-1,n}$ and $\mathfrak{M}^{l_0}_{m+1,n}$. First of all, let us recall that $\mathfrak{M}^{l_0}_{m\pm1,n}(\varphi,\epsilon,\theta,\tau,0,0)=e^{-n(\epsilon+i\varphi)}Z^{l_0}_{m\pm1,n}(\theta,\tau)$ and $\mathfrak{M}^{\dot{l}_0}_{\dot{m}\pm1,\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0)=e^{-\dot{n}(\epsilon-i\varphi)}Z^{\dot{l}_0}_{\dot{m}\pm1,\dot{n}}(\theta,\tau)$. Therefore,

$$\frac{\partial\mathfrak{M}^{l_0}_{m\pm1,n}}{\partial\varphi}=-in\mathfrak{M}^{l_0}_{m\pm1,n},\ \frac{\partial\mathfrak{M}^{l_0}_{m\pm1,n}}{\partial\epsilon}=-n\mathfrak{M}^{l_0}_{m\pm,n}\quad\text{and}$$

$$\times\frac{\partial\mathfrak{M}^{\dot{l}_0}_{\dot{m}\pm1,\dot{n}}}{\partial\varphi}=i\dot{n}\mathfrak{M}^{\dot{l}_0}_{\dot{m}\pm1,\dot{n}},\ \frac{\partial\mathfrak{M}^{\dot{l}_0}_{\dot{m}\pm1,\dot{n}}}{\partial\epsilon}=-\dot{n}\mathfrak{M}^{\dot{l}_0}_{\dot{m}\pm1,\dot{n}}.$$

Thus, in virtue of (57) the first bracket in (119) can be written as follows

$$i\,e^{-n(\epsilon+i\varphi)}\left[\frac{\partial Z^{l_0}_{m-1,n}}{\partial\theta}+i\frac{\partial Z^{l_0}_{m-1,n}}{\partial\tau}+\frac{2(n-(m-1)\cos\theta^c)}{\sin\theta^c}Z^{l_0}_{m-1,n}\right]$$

$$=2i\sqrt{(\dot{l}_0+\dot{m})(\dot{l}_0-\dot{m}+1)}\mathfrak{M}^{l_0}_{m,n}. \qquad (120)$$

Further, in virtue of (56) for the second and fourth brackets we have

$$i\,e^{-n(\epsilon+i\varphi)}\left[\frac{\partial Z^{l_0}_{m+1,n}}{\partial\theta}+i\frac{\partial Z^{l_0}_{m+1,n}}{\partial\tau}-\frac{2(n-(m+1)\cos\theta^c)}{\sin\theta^c}Z^{l_0}_{m+1,n}\right]$$

$$=2i\sqrt{(\dot{l}_0+\dot{m}+1)(\dot{l}_0-\dot{m})}\mathfrak{M}^{l_0}_{m,n}. \qquad (121)$$

Analogously, the third, fifth, and sixth brackets can be written as follows

$$-i\,e^{-n(\epsilon+i\varphi)}\left[\frac{\partial Z^{l_0}_{m-1,n}}{\partial\theta}+i\frac{\partial Z^{l_0}_{m-1,n}}{\partial\tau}+\frac{2(n-(m-1)\cos\theta^c)}{\sin\theta^c}Z^{l_0}_{m-1,n}\right]$$

$$=-2i\sqrt{(\dot{l}_0+\dot{m})(\dot{l}_0-\dot{m}+1)}\mathfrak{M}^{l_0}_{m,n},$$

$$-i\,e^{-n(\epsilon+i\varphi)}\left[\frac{\partial Z^{l_0}_{m+1,n}}{\partial\theta}+i\frac{\partial Z^{l_0}_{m+1,n}}{\partial\tau}-\frac{2(n-(m+1)\cos\theta^c)}{\sin\theta^c}Z^{l_0}_{m+1,n}\right]$$

$$=-2i\sqrt{(\dot{l}_0+\dot{m}+1)(\dot{l}_0-\dot{m})}\mathfrak{M}^{l_0}_{m,n}. \tag{122}$$

In like manner, the square brackets of the conjugate (dual) part of the system (119) can be rewritten as

$$-i\,e^{-\dot{n}(\epsilon-i\varphi)}\left[\frac{\partial Z^{\dot{l}_0}_{\dot{m}-1,\dot{n}}}{\partial\theta}-i\frac{\partial Z^{\dot{l}_0}_{\dot{m}-1,\dot{n}}}{\partial\tau}-\frac{2(\dot{n}-(\dot{m}-1)\cos\dot{\theta}^c)}{\sin\dot{\theta}^c}Z^{\dot{l}_0}_{\dot{m}-1,\dot{n}}\right]$$

$$=2i\sqrt{(l_0+m)(l_0-m+1)}\mathfrak{M}^{\dot{l}_0}_{\dot{m},\dot{n}} \tag{123}$$

for the seventh bracket (in virtue of the recurrence relation (59)) and

$$-i\,e^{-\dot{n}(\epsilon-i\varphi)}\left[\frac{\partial Z^{\dot{l}_0}_{\dot{m}+1,\dot{n}}}{\partial\theta}-i\frac{\partial Z^{\dot{l}_0}_{\dot{m}+1,\dot{n}}}{\partial\tau}+\frac{2(\dot{n}-(\dot{m}+1)\cos\dot{\theta}^c)}{\sin\dot{\theta}^c}Z^{\dot{l}_0}_{\dot{m}+1,\dot{n}}\right]$$

$$=2i\sqrt{(l_0+m+1)(l_0-m)}\mathfrak{M}^{\dot{l}_0}_{\dot{m},\dot{n}} \tag{124}$$

for the eighth and tenth brackets (in virtue of (58)). Finally, for the ninth, eleventh, and twelfth brackets we have

$$-i\,e^{-\dot{n}(\epsilon-i\varphi)}\left[\frac{\partial Z^{\dot{l}_0}_{\dot{m}-1,\dot{n}}}{\partial\theta}-i\frac{\partial Z^{\dot{l}_0}_{\dot{m}-1,\dot{n}}}{\partial\tau}-\frac{2(\dot{n}-(\dot{m}-1)\cos\dot{\theta}^c)}{\sin\dot{\theta}^c}Z^{\dot{l}_0}_{\dot{m}-1,\dot{n}}\right]$$

$$=\;-2i\sqrt{(l_0+m)(l_0-m+1)}\mathfrak{M}^{\dot{l}_0}_{\dot{m},\dot{n}},$$

$$-i\,e^{-\dot{n}(\epsilon-i\varphi)}\left[\frac{\partial Z^{\dot{l}_0}_{\dot{m}+1,\dot{n}}}{\partial\theta}-i\frac{\partial Z^{\dot{l}_0}_{\dot{m}+1,\dot{n}}}{\partial\tau}+\frac{2(\dot{n}-(\dot{m}+1)\cos\dot{\theta}^c)}{\sin\dot{\theta}^c}Z^{\dot{l}_0}_{\dot{m}+1,\dot{n}}\right]$$

$$=\;-2i\sqrt{(l_0+m+1)(l_0-m)}\mathfrak{M}^{\dot{l}_0}_{\dot{m},\dot{n}}. \tag{125}$$

Let us replace the square brackets in the system (119) via the relations (120)–(125) and cancel all the equations by $\mathfrak{M}^{l_0}_{mn}$ ($\mathfrak{M}^{\dot{l}_0}_{\dot{m}\dot{n}}$). In such a way, we see that the relativistically invariant system is reduced to a system of ordinary differential equations:

$$\sum_{k'} c^{kk'}_{l,l-1}\left[2\sqrt{l^2-m^2}\frac{d\boldsymbol{f}^{l_0}_{l-1,m,k'}(r)}{dr}-\frac{1}{r}(l+1)\sqrt{l^2-m^2}\,\boldsymbol{f}^{l_0}_{l-1,m,k'}(r)\right.$$
$$+\frac{i}{r}\sqrt{(1+m)(l+m-1)}\sqrt{(\dot{l}_0+\dot{m})(\dot{l}_0-\dot{m}+1)}\,\boldsymbol{f}^{l_0}_{l-1,m-1,k'}(r)$$
$$\left.+\ \frac{i}{r}\sqrt{(l-m)(l-m-1)}\sqrt{(\dot{l}_0+\dot{m}+1)(\dot{l}_0-\dot{m})}\,\boldsymbol{f}^{l_0}_{l-1,m+1,k'}(r)\right]$$
$$+\sum_{k'} c^{kk'}_{ll}\left[2m\frac{d\boldsymbol{f}^{l_0}_{l,m,k'}(r)}{dr}-\frac{1}{r}m\,\boldsymbol{f}^{l_0}_{l,m,k'}(r)\right.$$
$$-\frac{i}{r}\sqrt{(1+m)(l-m+1)}\sqrt{(\dot{l}_0+\dot{m})(\dot{l}_0-\dot{m}+1)}\,\boldsymbol{f}^{l_0}_{l,m-1,k'}(r)$$
$$\left.+\ \frac{i}{r}\sqrt{(l+m+1)(l-m)}\sqrt{(\dot{l}_0+\dot{m}+1)(\dot{l}_0-\dot{m})}\,\boldsymbol{f}^{l_0}_{l,m+1,k'}(r)\right]$$
$$+\sum_{k'} c^{kk'}_{l,l-1}\left[2\sqrt{(l+1)^2-m^2}\frac{d\boldsymbol{f}^{l_0}_{l+1,m,k'}(r)}{dr}-\frac{1}{r}\sqrt{(l+1)^2-m^2}\,\boldsymbol{f}^{l_0}_{l-1,m,k'}(r)\right.$$
$$-\frac{i}{r}\sqrt{(1-m)(l-m+2)}\sqrt{(\dot{l}_0+\dot{m})(\dot{l}_0-\dot{m}+1)}\,\boldsymbol{f}^{l_0}_{l+1,m-1,k'}(r)$$
$$\left.-\ \frac{i}{r}\sqrt{(l+m+1)(l+m+2)}\sqrt{(\dot{l}_0+\dot{m}+1)(\dot{l}_0-\dot{m})}\,\boldsymbol{f}^{l_0}_{l+1,m+1,k'}(r)\right]$$
$$+\kappa^c\,\boldsymbol{f}^{l_0}_{lmk}(r)=0,$$

$$\sum_{k'} c^{kk'}_{l,l-1}\left[2\sqrt{\dot{l}^2-\dot{m}^2}\frac{d\boldsymbol{f}^{l_0}_{l-1,\dot{m},\dot{k}'}(r)}{dr^*}-\frac{1}{r^*}(\dot{l}+1)\sqrt{l^2-\dot{m}^2}\,\boldsymbol{f}^{\dot{l}_0}_{\dot{l}-1,\dot{m},\dot{k}'}(r)\right.$$
$$+\frac{i}{r^*}\sqrt{(i+\dot{m})(\dot{l}+\dot{m}-1)}\sqrt{(l_0+m)(\dot{l}_0-\dot{m}+1)}\,\boldsymbol{f}^{\dot{l}_0}_{\dot{l}-1,\dot{m}-1,\dot{k}'}(r)$$
$$\left.+\ \frac{i}{r^*}\sqrt{(\dot{l}-\dot{m})(\dot{l}-\dot{m}-1)}\sqrt{(l_0+m+1)(l_0-m)}\,\boldsymbol{f}^{l_0}_{\dot{l}-1,\dot{m}+1,\dot{k}'}(r)\right]$$
$$+\sum_{\dot{k}'} c^{\dot{k}\dot{k}'}_{\dot{l}\dot{l}}\left[2\dot{m}\frac{d\boldsymbol{f}^{\dot{l}_0}_{\dot{l},\dot{m},\dot{k}'}(r)}{dr^*}-\frac{1}{r^*}\dot{m}\,\boldsymbol{f}^{\dot{l}_0}_{\dot{l},\dot{m},\dot{k}'}(r)\right.$$
$$-\frac{i}{r^*}\sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m}-1)}\sqrt{(l_0+m)(l_0-m+1)}\,\boldsymbol{f}^{\dot{l}_0}_{\dot{l},\dot{m}-1,\dot{k}'}(r)$$

$$+ \frac{i}{r^*}\sqrt{(\dot{l}+\dot{m}+1)(\dot{l}-\dot{m})}\sqrt{(l_0+m+1)(l_0-m)}\,\boldsymbol{f}^{l_0}_{\dot{l},\dot{m}+1,\dot{k}'}(r)\Bigg]$$

$$+ \sum_{\dot{k}'} c^{\dot{k}\dot{k}'}_{\dot{l},\dot{l}+1}\left[2\sqrt{(\dot{l}+1)^2-\dot{m}^2}\,\frac{d\,\boldsymbol{f}^{l_o}_{\dot{l}+1,\dot{m}\dot{k}'}}{dr^*}\right.$$

$$+ \frac{1}{r^*}\dot{l}\sqrt{(\dot{l}+1)^2-\dot{m}\,\boldsymbol{f}^{l_o}_{\dot{l}+1,\dot{m},\dot{k}'}}(r)$$

$$- \frac{i}{r^*}\sqrt{(\dot{l}-\dot{m}+1)(\dot{l}-\dot{m}+2)}\sqrt{(l_0+m)(l_0-m+1)}\,\boldsymbol{f}^{l_0}_{\dot{l}+1,\dot{m}-1,\dot{k}'}(r)$$

$$- \frac{i}{r^*}\sqrt{(\dot{l}+\dot{m}+1)(\dot{l}+\dot{m}+2)}\sqrt{(l_0+m)(l_0-m+1)}\,\boldsymbol{f}^{l_0}_{\dot{l}+1,\dot{m}-1,\dot{k}'}(r)$$

$$+ \dot{\kappa}^c\,\boldsymbol{f}^{l_0}_{\dot{l}\dot{m}\dot{k}'}(r) = 0.$$

4. DIRAC EQUATIONS

Let us consider general solutions of the Dirac equations

$$i\gamma_\mu \frac{\partial\psi}{\partial x_\mu} - m\psi = 0, \tag{126}$$

where we take γ-matrices in the Weyl basis:

$$\gamma_0 = \begin{pmatrix}\sigma_0 & 0\\ 0 & -\sigma_0\end{pmatrix},\quad \gamma_1 = \begin{pmatrix}0 & \sigma_1\\ -\sigma_1 & 0\end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix}0 & \sigma_2\\ -\sigma_2 & 0\end{pmatrix},\quad \gamma_3 = \begin{pmatrix}0 & \sigma_3\\ -\sigma_3 & 0\end{pmatrix}, \tag{127}$$

where

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

are the Pauli matrices. In accordance with a general Fermi-scheme of interlocking representations of $\mathfrak{G}_+$ (77), the equations (127) correspond to the following interlocking scheme

$$\left(\frac{1}{2},0\right) \longleftrightarrow \left(0,\frac{1}{2}\right)$$

By this reason a Dirac field is represented by a P-invariant direct sum $(1/2, 0) \oplus (0, 1/2)$ and the equations (127) describe a massive particle with the spin 1/2. It is easy to see that γ-matrices for the Dirac equations have the structure (81), where

$\underset{i}{\Delta} 2 = \sigma_i,\ \underset{i}{\overset{*}{\Delta}} 2 = -\sigma_i = \tilde{\sigma}_i$ Therefore, in accordance with (82) the Λ-matrices for the Dirac equations in bivector space are

$$\left.\begin{array}{l}\Lambda_1 = \sigma_1\sigma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \Lambda_2 = \sigma_2\sigma_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad \Lambda_3 = \sigma_3\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\\ \Lambda^*_1 = \tilde{\sigma}_2\tilde{\sigma}_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\quad \Lambda^*_2 = \tilde{\sigma}_3\tilde{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\quad \Lambda^*_3 = \tilde{\sigma}_1\tilde{\sigma}_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.\end{array}\right\} \tag{128}$$

It is easy to verify that matrices (129) satisfy the commutation relations (68)–(71) with infinitesimal operators A_i, B_i, $\tilde{\mathsf{A}}_i$, $\tilde{\mathsf{B}}_i$ defined for the fundamental representation of $\mathfrak{G}_+$ (see (104) and (109)).

It is obvious that for the representation $(1/2, 0) \oplus (0, 1/2)$ we can omit in the system (126) the index $k(k')$ which numerates subspaces transforming within one and the same representation of the group $\mathfrak{G}_+$. Further, assuming that $c\frac{1}{2}, -\frac{1}{2} = c\frac{1}{2}, \frac{1}{2} = 1$, $c\frac{1}{2}, \frac{3}{2} = 0$ we reduce the system (126) to the following

$$\begin{aligned}
&\frac{d\,\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r)}{dr} - \frac{1}{2r}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r) - \frac{i(\dot{l}+\frac{1}{2})}{r}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r) + \kappa^c\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r) = 0,\\
-&\frac{d\,\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r)}{dr} + \frac{1}{2r}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r) + \frac{i(\dot{l}+\frac{1}{2})}{r}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r) + \kappa^c\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r) = 0,\\
&\frac{d\,\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*)}{dr^*} - \frac{1}{2r^*}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*) - \frac{i(l+\frac{1}{2})}{r^*}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*) + \dot{\kappa}^c\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*) = 0,\\
-&\frac{d\,\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*)}{dr^*} + \frac{1}{2r^*}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*) + \frac{i(l+\frac{1}{2})}{r^*}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*) + \dot{\kappa}^c\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*) = 0,
\end{aligned} \tag{129}$$

where $\kappa^c = -im$. If we suppose that $\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r) = -i\,\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r)$ and $\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*) = -i\,\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r^*)$, then the system (130) takes a form

$$\begin{aligned}
&\frac{d\,\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r)}{dr} - \frac{(i+1)}{r}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r) + \kappa^c\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r) = 0,\\
&\frac{d\,\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r)}{dr} + \frac{i}{r}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r) - \kappa^c\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r) = 0,\\
&\frac{d\,\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r^*)}{dr^*} - \frac{(l+1)}{r^*}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*) + \dot{\kappa}^c\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r^*) = 0,
\end{aligned}$$

$$\frac{d\,\boldsymbol{f}^{\dot l}_{\frac12,\frac12}(r^*)}{dr^*}+\frac{l}{r^*}\boldsymbol{f}^{\dot l}_{\frac12,\frac12}(r^*)-\dot\kappa^c\boldsymbol{f}^{l}_{\frac12,\frac12}(r^*)=0,$$

Multiplying the second and fourth equations of the obtained system by 2 and adding the first equation with the second equation and the third equation with the fourth we come to the following differential equations

$$3\frac{d\,\boldsymbol{f}^{l}_{\frac12,\frac12}(r)}{dr}+\frac{\dot l-1}{r}\boldsymbol{f}^{l}_{\frac12,\frac12}(r)-\kappa^c\boldsymbol{f}^{l}_{\frac12,\frac12}(r)=0,$$
$$3\frac{d\,\boldsymbol{f}^{\dot l}_{\frac12,\frac12}(r^*)}{dr^*}+\frac{l-1}{r^*}\boldsymbol{f}^{\dot l}_{\frac12,\frac12}(r^*)-\dot\kappa^c\boldsymbol{f}^{\dot l}_{\frac12,\frac12}(r^*)=0. \quad (130)$$

Solutions of the equations (131) are expressed via Bessel functions of the half-integer order:

$$\boldsymbol{f}^{l}_{\frac12,\frac12}(r)=\sqrt[3]{r}\sum^{\infty}_{\kappa=0}(-1)^{\kappa}\left(\frac{2}{\sqrt3}\right)^{2\kappa}\Gamma(\dot\nu+k+1)J_{\dot\nu}\left(2\sqrt{\kappa^c}\sqrt[3]{r}\right),$$
$$\boldsymbol{f}^{\dot l}_{\frac12,\frac12}(r^*)=\sqrt[3]{r^*}\sum^{\infty}_{\kappa=0}(-1)^{\kappa}\left(\frac{2}{\sqrt3}\right)^{2\kappa}\Gamma(\nu+k+1)J_{\nu}\left(2\sqrt{\dot\kappa^c}\sqrt[3]{r}\right), \quad (131)$$

where $\dot\nu=-(\dot l-1)$, $\nu=-(l-1)$ and $\dot l=\frac{2\dot s+1}{2}$, $l=\frac{2s+1}{2}s=0,1,2,\ldots.$ The Bessel function in (132) has a form

$$J_{\frac{2s+1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\left[\sin\left(z-\frac{s\pi}{2}\right)\sum^{\frac{s}{2}}_{\kappa=0}\frac{(-1)^k(s+2\kappa)!}{(2\kappa)!(s-2\kappa)!(2z)^2k}\right.$$
$$\left.+\cos\left(z-\frac{s\pi}{2}\right)\sum^{[\frac{s-1}{2}]}_{k=0}\frac{(-1)^k(s+2k+1)!}{(2k+1)!(s-2k-1)!(2z)^{2k+1}}\right]. \quad (132)$$

In such a way, solutions of the Dirac equations have the form

$$\psi_1(r,\varphi^c,\theta^c)=\boldsymbol{f}^{\dot l}_{\frac12,\frac12}(r)\mathfrak{M}^{l}_{\frac12,n}(\varphi,\epsilon,\theta,\tau,0,0),$$
$$\psi_2(r,\varphi^c,\theta^c)=-i\,\boldsymbol{f}^{l}_{\frac12,\frac12}(r)\mathfrak{M}^{l}_{\frac12,n}(\varphi,\epsilon,\theta,0,0),$$
$$\dot\psi_1(r^*,\dot\varphi^c,\dot\theta^c)=\boldsymbol{f}^{\dot l}_{\frac12,\frac12}(r^*)\mathfrak{M}^{l}_{\frac12,\dot n}(\varphi,\epsilon,\theta,\tau,0,0),$$
$$\dot\psi_2(r^*,\dot\varphi^c,\dot\theta^c)=i\,\boldsymbol{f}^{\dot l}_{\frac12,\frac12}(r^*)\mathfrak{M}^{l}_{\frac12,\dot n}(\varphi,\epsilon,\theta,\tau,0,0),$$

where

$$l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}; \quad n = -l, -l+1, \dots, l;$$
$$\dot{l} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}; \quad \dot{n} = -\dot{l}, -\dot{l}+1, \dots, \dot{l},$$

$$\mathfrak{M}^{l}_{\pm\frac{1}{2},n}(\varphi, \epsilon, \theta, \tau, 0, 0) = e^{\mp\frac{1}{2}(\epsilon+i\varphi)} Z^{l}_{\pm\frac{1}{2},n}(\theta, \tau),$$

$$Z^{l}_{\pm\frac{1}{2},n}(\theta, \tau) = \cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum_{\kappa=-l}^{l} i^{\pm\frac{1}{2}-k}\tan^{\pm\frac{1}{2}-k}\frac{\theta}{2}\tanh^{n-k}\frac{\tau}{2}$$
$$\times\ {}_2F_1\left(\begin{matrix}\pm\frac{1}{2}-l+1, 1-l-k\\ \pm\frac{1}{2}-k+1\end{matrix}\middle| i^2\tan^2\frac{\theta}{2}\right)$$
$$\times\ {}_2F_1\left(\begin{matrix}n-l+1, 1-l-k\\ n-k+1\end{matrix}\middle| \tanh^2\frac{\tau}{2}\right),$$

$$\mathfrak{M}^{l}_{\pm\frac{1}{2},\dot{n}}(\varphi, \epsilon, \theta, \tau, 0, 0) = e^{\mp\frac{1}{2}(\epsilon+i\varphi)} Z^{i}_{\pm\frac{1}{2},n}(\theta, \tau),$$

$$Z^{\dot{l}}_{\pm\frac{1}{2},\dot{n}}(\theta, \tau) = \cos^{2i}\frac{\theta}{2}\cosh^{2i}\frac{\tau}{2}\sum_{k=-l}^{i} i^{\pm\frac{1}{2}-\dot{k}}\tan^{\pm\frac{1}{2}-\dot{k}}\frac{\theta}{2}\tanh^{\dot{n}-\dot{k}}\frac{\tau}{2}$$
$$\times\ {}_2F_1\left(\begin{matrix}\pm\frac{1}{2}-\dot{l}+1, 1-\dot{l}-\dot{k}\\ \pm\frac{1}{2}-\dot{k}+1\end{matrix}\middle| i^2\tan^2\frac{\theta}{2}\right)$$
$$\times\ {}_2F_1\left(\begin{matrix}\dot{n}-\dot{l}+1, 1-\dot{l}-\dot{k}\\ \dot{n}-\dot{k}+1\end{matrix}\middle| \tanh^2\frac{\tau}{2}\right).$$

5. WEYL EQUATIONS

Let us consider massless Dirac equations

$$i\gamma_\mu\frac{\partial\psi}{\partial x_\mu} = 0, \tag{133}$$

where γ-matrices have the form (128). Introducing projection operators $P_\pm = (1+\gamma_5)/2$ and following to the standard procedure (see, for example, Bogoliubov and Shirkov, 1993) we can split the equations (134) into two (Weyl) equations

$$\frac{\partial\phi_1}{\partial x_0} - \boldsymbol{\sigma}\frac{\partial\phi_1}{\partial \boldsymbol{x}} = 0, \qquad \frac{\partial\phi_2}{\partial x_0} - \boldsymbol{\sigma}\frac{\partial\phi_2}{\partial \boldsymbol{x}} = 0, \tag{134}$$

where

$$\phi_1 = \frac{1}{2}\begin{pmatrix}\psi_1 - \psi_3\\ \psi_2 - \psi_4\end{pmatrix}, \qquad \phi_2 = \frac{1}{2}\begin{pmatrix}\psi_1 + \psi_3\\ \psi_2 + \psi_4\end{pmatrix}.$$

As known, the equations (135) describe the massless particle with the spin 1/2 (neutrino).

Let us find solutions of the equations (134). In this case, taking into account (129) and (126), we come to a following system

$$\frac{d\,\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r)}{dr} - \frac{1}{2r}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r) - \frac{i\left(\dot{l}+\frac{1}{2}\right)}{r}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r) = 0,$$

$$-\frac{d\,\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r)}{dr} + \frac{1}{2r}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r) + \frac{i\left(\dot{l}+\frac{1}{2}\right)}{r}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r) = 0,$$

$$\frac{d\,\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*)}{dr^*} - \frac{1}{2r^*}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*) - \frac{i\left(l+\frac{1}{2}\right)}{r^*}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*) = 0,$$

$$-\frac{d\,\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*)}{dr^*} + \frac{1}{2r^*}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(r^*) + \frac{i\left(l+\frac{1}{2}\right)}{r^*}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*) = 0. \qquad (135)$$

As in the case of the Dirac equations we put $\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r) = -i\,\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r)$, $\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*) = -i\,\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*)$, and reduce the system (136) to the following equations

$$3\frac{d\,\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r)}{dr} + \frac{\dot{l}-1}{r}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r) = 0,$$

$$3\frac{d\,\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*)}{dr^*} + \frac{l-1}{r^*}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*) = 0. \qquad (136)$$

Solutions of the equations (137) are

$$\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r) = C\sqrt[3]{r}\sum_{k=0}^{\infty}(-1)^k\left(\frac{2}{\sqrt{3}}\right)^{2k}\Gamma(k+1)\Gamma(\dot{\nu}+k+1)J_{\dot{\nu}}\left(2\sqrt[3]{r}\right),$$

$$\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*) = \dot{C}\sqrt[3]{r^*}\sum_{k=0}^{\infty}(-1)^k\left(\frac{2}{\sqrt{3}}\right)^{2k}\Gamma(k+1)\Gamma(\nu+k+1)J_{\nu}\left(2\sqrt[3]{r^*}\right),$$

where $\dot{\nu} = -(\dot{l}-1)$, $\nu = -(l-1)$ and the Bessel function has the form (133).

Thus, solutions of the Weyl equations (135) are represented by linear combinations of the solutions of the massless Dirac equations (134):

$$\phi_1 = \frac{1}{2}\begin{pmatrix}\psi_1 - \dot{\psi}_1\\ \psi_2 - \dot{\psi}_2\end{pmatrix}, \qquad \phi_2 = \frac{1}{2}\begin{pmatrix}\psi_1 + \dot{\psi}_1\\ \psi_2 + \dot{\psi}_2\end{pmatrix},$$

where

$$\psi_1(r,\varphi^c,\theta^c)=\boldsymbol{f}^l_{\frac{1}{2},\frac{1}{2}}(r)\mathfrak{M}^l_{\frac{1}{2},n}(\varphi,\epsilon,\theta,\tau,0,0),$$
$$\psi_2(r,\varphi^c,\theta^c)=-i\,\boldsymbol{f}^l_{\frac{1}{2},\frac{1}{2}}(r)\mathfrak{M}^l_{-\frac{1}{2},n}(\varphi,\epsilon,\theta,0,0),$$
$$\dot{\psi}_1(r^*,\dot{\varphi}^c,\dot{\theta}^c)=\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*)\mathfrak{M}^l_{\frac{1}{2},\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0),$$
$$\dot{\psi}_2(r^*,\dot{\varphi}^c,\dot{\theta}^c)=-i\,\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*)\mathfrak{M}^l_{\frac{1}{2},\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0),$$

and

$$l=\frac{1}{2},\frac{3}{2},\frac{5}{2};\quad n=-l,-l+1,\ldots,l;$$
$$\dot{l}=\frac{1}{2},\frac{3}{2},\frac{5}{2};\quad \dot{n}=-\dot{l},-\dot{l}+1,\ldots,\dot{l},$$

$$\mathfrak{M}^l_{\pm\frac{1}{2},n}(\varphi,\epsilon,\theta,\tau,0,0)=e^{\mp\frac{1}{2}(\epsilon+i\varphi)}Z^l_{\pm\frac{1}{2},n}(\theta,\tau),$$

$$Z^l_{\pm\frac{1}{2},n}(\theta,\tau)=\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum_{k=-\dot{l}}^{l}i^{\pm\frac{1}{2}-k}\tan^{\pm\frac{1}{2}-k}\frac{\theta}{2}\tanh^{n-k}\frac{\tau}{2}$$
$$\times{}_2F_1\left(\begin{matrix}\pm\frac{1}{2}-l+1,1-\dot{l}-k\\ \pm\frac{1}{2}-k+1\end{matrix}\middle|\,i^2\tan^2\frac{\theta}{2}\right)$$
$$\times{}_2F_1\left(\begin{matrix}\dot{n}-\dot{l}+1,1-\dot{l}-k\\ n-k+1\end{matrix}\middle|\,\tanh^2\frac{\tau}{2}\right),$$

$$\mathfrak{M}^{\dot{l}}_{\pm\frac{1}{2},\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0)=e^{\mp\frac{1}{2}(\epsilon+i\varphi)}Z^{\dot{l}}_{\pm\frac{1}{2},\dot{n}}(\theta,\tau),$$

$$Z^{\dot{l}}_{\pm\frac{1}{2},\dot{n}}(\theta,\tau)=\cos^{2\dot{l}}\frac{\theta}{2}\cosh^{2\dot{l}}\frac{\tau}{2}\sum_{k=-l}^{\dot{l}}i^{\pm\frac{1}{2}-\dot{k}}\tan^{\pm\frac{1}{2}-\dot{k}}\frac{\theta}{2}\tanh^{\dot{n}-\dot{k}}\frac{\tau}{2}$$
$$\times{}_2F_1\left(\begin{matrix}\pm\frac{1}{2}-\dot{l}+1,1-\dot{l}-\dot{k}\\ \pm\frac{1}{2}-\dot{k}+1\end{matrix}\middle|\,i^2\tan^2\frac{\theta}{2}\right)$$
$$\times{}_2F_1\left(\begin{matrix}\dot{n}-\dot{l}+1,1-\dot{l}-\dot{k}\\ \dot{n}-\dot{k}+1\end{matrix}\middle|\,\tanh^2\frac{\tau}{2}\right).$$

6. MAXWELL EQUATIONS

In accordance with a general Bose-scheme of the interlocking representations of the group $\mathfrak{G}_+$ (76), the Maxwell field (1, 0) ⊕ (0, 1) corresponds to a following interlocking scheme

$$(1,0) \longleftrightarrow \left(\frac{1}{2},\frac{1}{2}\right) \longleftrightarrow (0,1)$$

By this reason and in agreement with a de Broglie theory of fusion (de Brolgie, 1943) the Maxwell field (1, 0) ⊕ (0, 1) is defined in terms of tensor products (1/2, 0) ⊗ (1/2, 0) and (0, 1/2) ⊗ (0, 1/2). Indeed, according to (6) the Clifford algebras, corresponded to the Maxwell fields, are $\mathbb{C}_2 \otimes \mathbb{C}_2$ and $\overset{*}{\mathbb{C}}_2 \otimes \overset{*}{\mathbb{C}}_2$. The algebras $\mathbb{C}_2 \otimes \mathbb{C}_2$ and $\overset{*}{\mathbb{C}}_2 \otimes \overset{*}{\mathbb{C}}_2$ induce spinspaces $\mathbb{S}_2 \otimes \mathbb{S}_2$ and $\dot{\mathbb{S}}_2 \otimes \dot{\mathbb{S}}_2$. These spinspaces are full representation spaces for tensor products $\boldsymbol{\tau}_{\frac{1}{2},0} \otimes \boldsymbol{\tau}_{\frac{1}{2},0}$ and $\boldsymbol{\tau}_{0,\frac{1}{2}} \otimes \boldsymbol{\tau}_{0,\frac{1}{2}}$. In accordance with (8) the basis "vectors" (spintensors) of the spinspaces $\mathbb{S}_2 \otimes \mathbb{S}_2$ and $\dot{\mathbb{S}}_2 \otimes \dot{\mathbb{S}}_2$ have the form

$$\begin{aligned} &\xi^{11} = \xi^1 \otimes \xi^1, \quad \xi^{12} = \xi^1 \otimes \xi^2, \quad \xi^{21} = \xi^2 \otimes \xi^1, \quad \xi^{22} = \xi^2 \otimes \xi^2, \\ &\xi^{\dot{1}\dot{1}} = \xi^{\dot{1}} \otimes \xi^{\dot{1}}, \quad \xi^{\dot{1}\dot{2}} = \xi^{\dot{1}} \otimes \xi^{\dot{2}}, \quad \xi^{\dot{2}\dot{1}} = \xi^{\dot{2}} \otimes \xi^{\dot{1}} \quad \xi^{\dot{2}\dot{2}} = \xi^{\dot{2}} \otimes \xi^{\dot{2}}. \end{aligned} \tag{137}$$

Further, the representations $\boldsymbol{\tau}_{\frac{1}{2},0} \otimes \boldsymbol{\tau}_{\frac{1}{2},0}$ and $\boldsymbol{\tau}_{0,\frac{1}{2}} \otimes \boldsymbol{\tau}_{0,\frac{1}{2}}$ are reducible. In virtue of the Clebsh–Gordan formula

$$\boldsymbol{\tau}_{l_1,\dot{l}_1} \otimes \boldsymbol{\tau}_{l_2,\dot{l}_2} = \sum_{|l_1-l_2|\le k\le l_1+l_2;|\dot{l}_1-\dot{l}_2|\le\dot{k}\le\dot{k}\le\dot{l}_1+\dot{l}_2} \boldsymbol{\tau}_{k,\dot{k}}$$

we have

$$\begin{aligned} \boldsymbol{\tau}_{\frac{1}{2},0} \otimes \boldsymbol{\tau}_{\frac{1}{2},0} &= \boldsymbol{\tau}_{0,0} \oplus \boldsymbol{\tau}_{0,1}, \\ \boldsymbol{\tau}_{0,\frac{1}{2}} \otimes \boldsymbol{\tau}_{0,\frac{1}{2}} &= \boldsymbol{\tau}_{0,0} \oplus \boldsymbol{\tau}_{0,1}. \end{aligned}$$

At this point the spinspaces $\mathbb{S}_2 \otimes \mathbb{S}_2$ and $\dot{\mathbb{S}}_2 \otimes \dot{\mathbb{S}}_2$ decompose into direct sums of the symmetric representation spaces:

$$\begin{aligned} \mathbb{S}_2 \otimes \mathbb{S}_2 &= \mathrm{Sym}_{(0,0)} \oplus \mathrm{Sym}_{(2,0)}, \\ \dot{\mathbb{S}}_2 \otimes \dot{\mathbb{S}}_2 &= \mathrm{Sym}_{(0,0)} \oplus \mathrm{Sym}_{(0,2)}. \end{aligned}$$

The spintensors ξ^{11}, $\xi^{12} = \xi^{21}$, ξ^{22} and $\xi^{\dot{1}\dot{1}}$, $\xi^{\dot{1}\dot{2}} = \xi^{\dot{2}\dot{1}}$, $\xi^{\dot{2}\dot{2}}$, obtained after symmetrization from (138) compose the basises of three-dimensional complex spaces $\mathrm{Sym}_{(2,0)}$ and $\mathrm{Sym}_{(0,2)}$, respectively. Let us introduce independent complex coordinates F_1, F_2, F_3 and $\overset{*}{F}_1$, $\overset{*}{F}_2$, $\overset{*}{F}_3$ for the spintensors $f^{\lambda\mu}$ and $f^{\dot{\lambda}\dot{\mu}}$ (spinor representations of the electromagnetic tensor), where $F_i = E_i - iB_i$ and $\overset{*}{F}_i = E_i + iB_i$. Explicit expressions of the spinor representations of the electromagnetic tensor are

$$\boldsymbol{\tau}_{1,0}\begin{cases} f^{11} \sim 4(F_1 + iF_2), \\ f^{12} \sim 4F_3, \\ f^{22} \sim 4(F_1 + iF_2); \end{cases}$$

$$\boldsymbol{\tau}_{0,1}\begin{cases} f^{\dot{1}\dot{1}} \sim 4\big(\overset{*}{F}_1 + i\overset{*}{F}_2\big), \\ f^{\dot{1}\dot{2}} \sim 4\overset{*}{F}_3, \\ f^{\dot{2}\dot{2}} \sim 4\big(\overset{*}{F}_1 + i\overset{*}{F}_2\big). \end{cases}$$

In such a way, we see that complex linear combinations $\mathbf{F} = \mathbf{E} - i\mathbf{B}$ and $\overset{*}{F} = \mathbf{E} + i\mathbf{B}$, transformed within $\boldsymbol{\tau}_{1,0}$ and $\boldsymbol{\tau}_{0,1}$ representations, are coincide with the Majorana–Oppenheimer wave functions. At the beginning of 30s of the last century Majorana (unpublished) and Oppenheimer (1931) proposed to consider the Maxwell theory of electromagnetism as the wave mechanics of the photon. They introduced a wave function of the form

$$\psi = \mathbf{E} - i\mathbf{B}, \tag{138}$$

where $\mathbf{E}$ and $\mathbf{B}$ are electric and magnetic fields. In virtue of this the standard Maxwell equations can be rewritten

$$\text{div}\psi = \rho, \qquad i\,\text{rot}\psi = \dot{\text{j}} + \frac{\partial\psi}{\partial t}, \tag{139}$$

where $\psi = (\psi_1, \psi_2, \psi_3)$, $\psi_k = E_k - iB_k (k = 1, 2, 3)$. In accordance with correspondence principle $(-i\partial/\partial x_i \rightarrow p_i; +i\partial/\partial t \rightarrow W)$ and in absence of electric charges and currents the equations (140) take a Dirac-like form

$$(W - \boldsymbol{\alpha}\cdot\boldsymbol{p})\psi = 0 \tag{140}$$

with transversality condition

$$\boldsymbol{p}\cdot\boldsymbol{\psi} = 0.$$

At this point three matrices

$$\alpha^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{141}$$

satisfy the angular-momentum commutation rules

$$[\alpha_i, \alpha_k] = -i\varepsilon_{ikl}\alpha_l \quad (i, k, l = 1, 2, 3).$$

Further, for the complex conjugate wave function

$$\psi^* = \mathbf{E} + i\mathbf{B} \tag{142}$$

there are the analogous Dirac-like equations

$$(W + \boldsymbol{\alpha}\cdot\boldsymbol{p})\psi^* = 0. \tag{143}$$

In such a way, ψ (ψ^*) may be considered as a wave function of the photon satisfying the massless Dirac-like equations (Maxwell equations in the Dirac form are considered also in (Moses, 1958, 1959)). In contrast to the Gupta–Bleuler phenomenology, where the nonobservable four-potential A_μ is quantized, the main advantage of the Majorana–Oppenheimer formulation of electrodynamics lies in the fact that it deals directly with observable quantities, such as the electric and magnetic fields (Esposito, 1998; Giannetto, 1985; Recami, 1990; Varlamov, 2002a,c).

It is easy to see that Maxwell equations in the Majorana–Oppenheimer form can be mapped into the bivector space $\mathbb{R}^6$. At this point, for the Λ-matrices we have

$$\left.\begin{array}{l}
\Lambda_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_1^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Lambda_2^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \Lambda_3^* = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{array}\right\} \tag{144}$$

Infinitesimal operators in the representation spaces $\mathrm{Sym}_{(2,0)}$ and $\mathrm{Sym}_{(0,2)}$ take a form

$$\left.\begin{array}{l}
\mathsf{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathsf{A}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \mathsf{A}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\mathsf{B}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \mathsf{B}_2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \quad \mathsf{B}_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\tilde{\mathsf{A}}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \tilde{\mathsf{A}}_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathsf{A}}_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\tilde{\mathsf{B}}_1 = s\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix}, \quad \tilde{\mathsf{B}}_2 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \tilde{\mathsf{B}}_3 = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\end{array}\right\} \tag{145}$$

It is easy to verify that the matrices (145) and operators (146) satisfy the commutation relations (68)–(71).

In the case of spin 1 the system (126) takes a form

$$
\begin{aligned}
&2\frac{d\boldsymbol{f}^{l}_{1,1}(r)}{dr}-\frac{1}{r}\boldsymbol{f}^{l}_{1,1}(r)-\frac{i\sqrt{2\dot{l}(\dot{l}+1)}}{r}\boldsymbol{f}^{l}_{1,0}(r)+\frac{i\sqrt{2\dot{l}(\dot{l}+1)}}{r}\boldsymbol{f}^{l}_{0,0}(r)=0,\\
&2\frac{d\boldsymbol{f}^{l}_{0,0}(r)}{dr}-\frac{2}{r}\boldsymbol{f}^{l}_{0,0}(r)-\frac{i\sqrt{2\dot{l}(\dot{l}+1)}}{r}\boldsymbol{f}^{l}_{1,-1}(r)+\frac{i\sqrt{2\dot{l}(\dot{l}+1)}}{r}\boldsymbol{f}^{l}_{1,1}(r)=0,\\
&-2\frac{d\boldsymbol{f}^{l}_{1,-1}(r)}{dr}+\frac{1}{r}\boldsymbol{f}^{l}_{1,-1}(r)+\frac{i\sqrt{2\dot{l}(\dot{l}+1)}}{r}\boldsymbol{f}^{l}_{1,0}(r)+\frac{i\sqrt{2\dot{l}(\dot{l}+1)}}{r}\boldsymbol{f}^{l}_{0,0}(r)=0,\\
&2\frac{d\boldsymbol{f}^{\dot{l}}_{1,1}(r^*)}{dr}-\frac{1}{r^*}\boldsymbol{f}^{\dot{l}}_{1,1}(r^*)-\frac{i\sqrt{2l(l+1)}}{r^*}\boldsymbol{f}^{\dot{l}}_{1,0}(r^*)+\frac{i\sqrt{2l(l+1)}}{r}\boldsymbol{f}^{\dot{l}}_{0,0}(r^*)=0,\\
&2\frac{d\boldsymbol{f}^{\dot{l}}_{0,0}(r^*)}{dr}-\frac{2}{r^*}\boldsymbol{f}^{\dot{l}}_{0,0}(r^*)-\frac{i\sqrt{2l(l+1)}}{r^*}\boldsymbol{f}^{\dot{l}}_{1,-1}(r^*)+\frac{i\sqrt{2l(l+1)}}{r^*}\boldsymbol{f}^{\dot{l}}_{1,1}(r^*)=0,\\
&-2\frac{d\boldsymbol{f}^{\dot{l}}_{1,-1}(r^*)}{dr^*}+\frac{1}{r^*}\boldsymbol{f}^{\dot{l}}_{1,-1}(r^*)+\frac{i\sqrt{2l(l+1)}}{r^*}\boldsymbol{f}^{\dot{l}}_{1,0}(r^*)\\
&+\frac{i\sqrt{2l(l+1)}}{r^*}\boldsymbol{f}^{\dot{l}}_{0,0}(r^*)=0,
\end{aligned}
\qquad (146)
$$

It is easy to see that in the system (147) there are two superfluous components $\boldsymbol{f}^{l}_{0,0}(r)$ and $\boldsymbol{f}^{\dot{l}}_{0,0}(r^*)$. The requirement $\boldsymbol{f}^{l}_{0,0}(r)=\boldsymbol{f}^{\dot{l}}_{0,0}(r^*)=0$ gives rise to $\boldsymbol{f}^{l}_{1,-1}(r)=\boldsymbol{f}^{l}_{1,1}(r)$ and $\boldsymbol{f}^{\dot{l}}_{1,-1}(r^*)=\boldsymbol{f}^{\dot{l}}_{1,1}(r^*)$ (it follows from the second and fourth equations). Taking into account these relations we can rewrite the system (147) as follows

$$
\begin{aligned}
2\frac{d\boldsymbol{f}^{l}_{1,1}(r)}{dr}-\frac{1}{r}\boldsymbol{f}^{l}_{1,1}(r)-\frac{i\sqrt{2\dot{l}(\dot{l}+1)}}{r}\boldsymbol{f}^{l}_{1,0}(r)&=0,\\
-2\frac{d\boldsymbol{f}^{l}_{1,-1}(r)}{dr}+\frac{1}{r}\boldsymbol{f}^{l}_{1,-1}(r)+\frac{i\sqrt{2\dot{l}(\dot{l}+1)}}{r}\boldsymbol{f}^{l}_{1,0}(r)&=0,\\
2\frac{d\boldsymbol{f}^{\dot{l}}_{1,1}(r^*)}{dr^*}-\frac{1}{r^*}\boldsymbol{f}^{\dot{l}}_{1,1}(r^*)-\frac{i\sqrt{2l(l+1)}}{r^*}\boldsymbol{f}^{\dot{l}}_{1,0}(r^*)&=0,\\
-2\frac{d\boldsymbol{f}^{\dot{l}}_{1,-1}(r^*)}{dr^*}+\frac{1}{r^*}\boldsymbol{f}^{\dot{l}}_{1,-1}(r^*)+\frac{i\sqrt{2l(l+1)}}{r^*}\boldsymbol{f}^{\dot{l}}_{1,0}(r^*)&=0,
\end{aligned}
\qquad (147)
$$

Solutions of the system (148) also can be expressed via the Bessel functions of integer order:

$$\boldsymbol{f}^{l}_{1,1}(r) = \boldsymbol{f}^{l}_{1,-1}(r) = c_1 \sum_{k=0}^{\infty}(-1)^k 2^{2k}\Gamma(k+1)\Gamma(-\dot{l}+k+1)(2\sqrt{r})^{\dot{l}-1} J_{-\dot{l}}(2\sqrt{r}),$$

$$\boldsymbol{f}^{l}_{1,0}(r) = \frac{c_2}{\sqrt{2\dot{l}(\dot{l}+1)}} \sum_{k=0}^{\infty}(-1)^k(2k-3)2^{2k}\Gamma(k+1)\Gamma(-\dot{l}+k+1) \times (2\sqrt{r})^{\dot{l}} J_{-\dot{l}}(2\sqrt{r}),$$

$$\boldsymbol{f}^{\dot{l}}_{1,1}(r^*) = \boldsymbol{f}^{\dot{l}}_{1,-1}(r^*) = \dot{c}_1 \sum_{k=0}^{\infty}(-1)^k 2^{2k}\Gamma(k+1)\Gamma(-l+k+1)(2\sqrt{r^*})^{l-1} \times J_{-l}(2\sqrt{r^*}),$$

$$\boldsymbol{f}^{\dot{l}}_{1,0}(r^*) = \frac{\dot{c}_2}{\sqrt{2l(l+1)}} \sum_{k=0}^{\infty}(-1)^k(2k-3)2^{2k}\Gamma(k+1)\Gamma(-l+k+1) \times (2\sqrt{r^*})^{l} J_{-l}(2\sqrt{r^*}),$$

where

$$J_{-\nu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(-\nu+k+1)} \left(\frac{z}{2}\right)^{-\nu+2k}.$$

In such a way, solutions of the Maxwell equations in vacuum are

$$\psi_1(r, \varphi^c, \theta^c) = \boldsymbol{f}^{l}_{1,1}(r)\mathfrak{M}^{l}_{1,n}(\varphi, \epsilon, \theta, \tau, 0, 0),$$

$$\psi_2(r, \varphi^c, \theta^c) = \boldsymbol{f}^{l}_{1,0}(r)\mathfrak{M}^{l}_{0,n}(\varphi, \epsilon, \theta, \tau, 0, 0),$$

$$\psi_3(r, \varphi^c, \theta^c) = \boldsymbol{f}^{l}_{1,-1}(r)\mathfrak{M}^{l}_{-1,n}(\varphi, \epsilon, \theta, \tau, 0, 0),$$

$$\dot{\psi}_1(r^*, \dot{\varphi}^c, \dot{\theta}^c) = \boldsymbol{f}^{\dot{l}}_{1,1}(r^*)\mathfrak{M}^{\dot{l}}_{1,\dot{n}}(\varphi, \epsilon, \theta, \tau, 0, 0),$$

$$\dot{\psi}_2(r^*, \dot{\varphi}^c, \dot{\theta}^c) = \boldsymbol{f}^{\dot{l}}_{1,0}(r^*)\mathfrak{M}^{\dot{l}}_{1,\dot{n}}(\varphi, \epsilon, \theta, \tau, 0, 0),$$

$$\dot{\psi}_3(r^*, \dot{\varphi}^c, \dot{\theta}^c) = \boldsymbol{f}^{\dot{l}}_{1,-1}(r^*)\mathfrak{M}^{\dot{l}}_{1,\dot{n}}(\varphi, \epsilon, \theta, \tau, 0, 0),$$

where

$$l = 1, 2, 3; \quad n = -l, -l+1, \ldots, l;$$

$$\dot{l} = 1, 2, 3; \quad \dot{n} = -\dot{l}, -\dot{l}+1, \ldots, \dot{l},$$

$$\mathfrak{M}^{l}_{\pm 1,n}(\varphi, \epsilon, \theta, \tau, 0, 0) = e^{\mp(\epsilon+i\varphi)} Z^{l}_{\pm 1,n}(\theta, \tau),$$

$$
\mathsf{Z}^{l}_{\pm 1,n}(\theta,\tau)=\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum_{k=-l}^{l} i^{\pm\frac{1}{2}-k}\tan^{\pm 1-k}\frac{\theta}{2}\tanh^{n-k}\frac{\tau}{2}
\times {}_2F_1\left(\begin{matrix}\pm 1-l+1,\,1-l-k\\ \pm 1-k+1\end{matrix}\middle| i^2\tan^2\frac{\theta}{2}\right)
\times {}_2F_1\left(\begin{matrix}n-l+1,\,1-l-k\\ n-k+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right),
$$

$$
\mathfrak{M}^{l}_{0,n},(\varphi,\epsilon,\theta,\tau,0,0)=Z^{l}_{0,n}(\theta,\tau),
$$

$$
\mathsf{Z}^{l}_{0,n}(\theta,\tau)=\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum_{k=-l}^{l} i^{-k}\tan^{-k}\frac{\theta}{2}\tanh^{n-k}\frac{\tau}{2}
\times {}_2F_1\left(\begin{matrix}-l+1,\,1-l-k\\ -k+1\end{matrix}\middle| i^2\tan^2\frac{\theta}{2}\right)
\times {}_2F_1\left(\begin{matrix}n-l+1,\,1-l-k\\ n-k+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right),
$$

$$
\mathfrak{M}^{\dot{l}}_{\pm 1,\dot{n}},(\varphi,\epsilon,\theta,\tau,0,0)=e^{\mp(\epsilon+i\varphi)}\mathsf{Z}^{\dot{l}}_{\pm 1,\dot{n}}(\theta,\tau),
$$

$$
\mathsf{Z}^{\dot{l}}_{0,\dot{n}}(\theta,\tau)=\cos^{2\dot{l}}\frac{\theta}{2}\cosh^{2\dot{l}}\frac{\tau}{2}\sum_{\dot{k}=-\dot{l}}^{\dot{l}} i^{\pm-\dot{k}}\tan^{\pm 1-\dot{k}}\frac{\theta}{2}\tanh^{\dot{n}-\dot{k}}\frac{\tau}{2}
\times {}_2F_1\left(\begin{matrix}\pm 1-\dot{l}+1,\,1-\dot{l}-\dot{k}\\ \pm 1-\dot{k}+1\end{matrix}\middle| i^2\tan^2\frac{\theta}{2}\right)
\times {}_2F_1\left(\begin{matrix}\dot{n}-\dot{l}+1,\,1-\dot{l}-\dot{k}\\ \dot{n}-\dot{k}+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right),
$$

$$
\mathfrak{M}^{\dot{l}}_{0,\dot{n}},(\varphi,\epsilon,\theta,\tau,0,0)=Z^{\dot{l}}_{0,\dot{n}}(\theta,\tau),
$$

$$
\mathsf{Z}^{\dot{l}}_{0,\dot{n}}(\theta,\tau)=\cos^{2\dot{l}}\frac{\theta}{2}\cosh^{2\dot{l}}\frac{\tau}{2}\sum_{\dot{k}=-\dot{l}}^{\dot{l}} i^{\pm-\dot{k}}\tan^{-\dot{k}}\frac{\theta}{2}\tanh^{\dot{n}-\dot{k}}\frac{\tau}{2}
\times {}_2F_1\left(\begin{matrix}-\dot{l}+1,\,1-\dot{l}-\dot{k}\\ -\dot{k}+1\end{matrix}\middle| i^2\tan^2\frac{\theta}{2}\right)
\times {}_2F_1\left(\begin{matrix}\dot{n}-\dot{l}+1,\,1-\dot{l}-\dot{k}\\ \dot{n}-\dot{k}+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right),
$$

Moving further on the general Bose- and Fermi-schemes of the interlocking representations (76) and (77), we can obtain in the same way solutions of all other higher-spin equations in terms of the functions on the Lorentz group.

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